

Cartesian Product of Sets.

Let A and B be two sets. The Cartesian Product of A and B is the set of all ordered pairs whose 1st coordinates belong to set A and second to set B .

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

Binary Relation.

A binary relation R from a set A to a set B is any subset of $A \times B$.

If $(a, b) \in R$ then a is related to b or aRb .

If $(a, b) \notin R$ then $a \not R b$.

Binary relation on a set A .

A binary relation on a set A is any subset of $A \times A$.

$$\emptyset \subseteq R \subseteq A \times A.$$

Remark.

$$(1) \quad (a, b) = (c, d) \Leftrightarrow a = c, b = d$$

$$|A \times B| = |A| \cdot |B|$$

$$(2) \quad A \times B \neq B \times A.$$

For example,

$$A = \{1, 2\}, B = \{a, b, c\}.$$

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$

$$\text{and } B \times A = \{(a, 1), (b, 1), (c, 1), (a, 2), (b, 2), (c, 2)\}.$$

$$\Rightarrow A \times B \neq B \times A.$$

(2)

Types of Relation.

① Reflexive relation.

A relation is said to be reflexive if $aRa \quad \forall a \in A$.

i.e. $(a,a) \in R \quad \forall a \in A$.

② Symmetric relation.

A relation is said to be symmetric if aRb then bRa . $\forall a, b \in A$.

i.e. $(a,b) \in R \Rightarrow (b,a) \in R \quad \forall a, b \in A$.

③ Anti-Symmetric relation.

A relation is said to be anti-symmetric if aRb and bRa .

$\Rightarrow a=b \quad \forall a, b \in A$.

i.e. $(a,b), (b,a) \in R \Rightarrow a=b \quad \forall a, b \in A$.

④ Transitive relation.

A relation is said to be transitive if aRb and bRc .

$\Rightarrow aRc \quad \forall a, b, c \in A$.

i.e. $(a,b), (b,c) \in R \Rightarrow (a,c) \in R$.

$\forall a, b, c \in A$.

Example.

$$A = \{1, 2, 3, 4\}$$

$$R = \{(1,1), (1,2), (2,3), (1,3), (4,4)\}$$

- (i) Not Reflexive (ii) Not symmetric
(iii) Anti symmetric (iv) Transitive.

نوٹ: اگر کسی property کی violation نہ ہو تو وہ properly true ہوگی۔
trivially true

(3)

Example.

Determine which of the following relations are (a) reflexive (b) symmetric (c) antisymmetric (d) transitive.

Solution.

(1) Relation $\leq$ on the set $\mathbb{Z}$ of integers.

(a) Reflexive, because $x \leq x$ for every $x \in \mathbb{Z}$.

(b) Not symmetric because $3 \leq 4$ but $4 \not\leq 3$.

(c) Antisymmetric, since whenever $a \leq b$ and $b \leq a \Rightarrow a = b$.

(d) Transitive, if $a \leq b$ and $b \leq c$ then $a \leq c$.

(2) Set inclusion $\subseteq$ on a collection G of sets.

(a) Reflexive, since $A \subseteq A$ for any set A in G .

(b) Not symmetric because $\{1, 2\} \subseteq \{1, 2, 3\}$ but $\{1, 2, 3\} \not\subseteq \{1, 2\}$.

(c) Antisymmetric, whenever $A \subseteq B$ and $B \subseteq A$ then $A = B$.

(d) Transitive, if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

(3) Relation $\perp$ on the set L of lines in the plane.

(a) Not reflexive, since no line is $\perp$ to itself.

(b) Symmetric, since if the line $a \perp b$ then $b \perp a$.

(c) Not antisymmetric, since we can have

(4)

distinct lines a and b such that $a \perp b$ and $b \perp a$.

(d) Not Transitive, if $a \perp b$ and $b \perp c$ then it is not true that $a \perp c$.

(4) Relation $\parallel$ (Parallel) on the set L of lines in the plane.

(a) Not reflexive, since no line is parallel to itself.

Note: There is no common point in the intersection of two parallel lines. If we take one line then it has common points.

(b) Symmetric, if $a \parallel b$ then $b \parallel a$.

(c) Not antisymmetric, since we can have distinct lines a and b such that $a \parallel b$ and $b \parallel a$.

(d) Not transitive, if $a \parallel b$ and $b \parallel c$ then it is not true that $a \parallel c$.

(5) Relation " $\mid$ " of divisibility on the set P of +ive integers.

(a) Reflexive, since $n \mid n$ for every +ive integer n in P .

(b) Not Symmetric, because $2 \mid 6$ but $6 \nmid 2$.

(c) Antisymmetric, since whenever $m \mid n$ and $n \mid m$ then $m = n$.

to $3 \mid -3$ and $-3 \mid 3$ but $3 \neq -3$.

(5)

(d) If a/b and b/c then a/c , (5) is transitive.

Example.

$$X = \{1, 2, 3\}, R_1 = \{(1, 1), (2, 2), (3, 3)\}$$

Symmetric as well as antisymmetric.

$$R_2 = \{(1, 1), (2, 3), (3, 2)\}$$

Symmetric but not antisymmetric.

$$R_3 = \{(1, 1), (2, 2), (2, 3)\}$$

Antisymmetric but not symmetric.

Partition.

Let S be a non empty set. A partition of S is a subdivision of S into non overlapping non empty subsets i.e. it is a collection

$P = \{A_i\}$ of non empty subsets of S such that

(i) each $a \in S$ belong to one of A_i ; $S = \cup A_i$

(ii) $A_i \cap A_j = \emptyset$ for $i \neq j$ (the sets are mutually disjoint)

The subsets A_i are also called cells.

Examples.

① $\mathbb{Z}$ the set of integers ② E the set of even integers ③ the set of Odd numbers = $\emptyset$.

The $\{E, O\}$ defines partition in $\mathbb{Z}$.

$$S = \{1, 2, 3, \dots, 10\}$$

$$P_1 = \{\{1, 3, 5\}, \{2, 6\}, \{4, 8, 9\}\}$$

$$7 \notin A_i \quad \forall i$$

$\Rightarrow P_1$ does not define partition. (1st axiom fail).

(6)

$$P_2 = \{\{1, 3, 5\}, \{2, 4, 6, 8\}, \{5, 7, 9\}\}$$

$\Rightarrow P_2$ does not define partition. (2nd axiom for

Representative of the class.

Given a partition $P = \{A_i\}$ of a set S , any element $b \in A_i$ is called representative of the cell (class) A_i and a subset B is called system of representatives if B contains exactly one element of each of the cell.

Equivalence relation.

Let S be a non empty set. A relation R on S is an equivalence relation if it has following properties.

(i) R is reflexive. (ii) R is symmetric.

(iii) R is transitive.

Example.

Write all possible partitions of $X = \{a, b, c, d\}$.

Solution.

$$P_1 = \{a, b, c, d\} = X; \quad P_2 = \{\{a\}, \{b, c, d\}\}$$

$$P_3 = \{\{b\}, \{a, c, d\}\}; \quad P_4 = \{\{c\}, \{a, b, d\}\}$$

$$P_5 = \{\{d\}, \{a, b, c\}\}; \quad P_6 = \{\{a, b\}, \{c, d\}\}$$

$$P_7 = \{\{a, c\}, \{b, d\}\}; \quad P_8 = \{\{a, d\}, \{b, c\}\}$$

$$P_9 = \{\{a\}, \{b\}, \{c\}, \{d\}\}; \quad P_{10} = \{\{a\}, \{b\}, \{c, d\}\}$$

$$\{\{a, c\}, \{b\}, \{d\}\}; \quad P_{12} = \{\{a\}, \{c\}, \{b, d\}\}$$

(7)

$$P_{13} = \{\{a\}, \{d\}, \{b, c\}\} ; P_{14} = \{\{b\}, \{c\}, \{a, d\}\}$$

$$P_5 = \{\{c\}, \{d\}, \{a, b\}\}$$

Note:

Let m be a fixed +ive integer. Two integers a and b are said to be Congruent modulo m , $a \equiv b \pmod{m}$ iff $m \mid a-b$.

Show that this relation is an equivalence relation.

Since $n \mid 0$. Therefore, $n \mid a-a$.

$$\Rightarrow a \equiv a \pmod{m}$$

$\Rightarrow \equiv$ is reflexive.

Let $a \equiv b \pmod{m}$. Then $m \mid a-b$.

$$\text{Now, } b-a = -(a-b)$$

$$\Rightarrow m \mid b-a$$

$$\Rightarrow b \equiv a \pmod{m}$$

$\Rightarrow \equiv$ is symmetric.

Let $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$.

$$\Rightarrow m \mid a-b \quad \text{and} \quad m \mid b-c$$

$$\Rightarrow m \mid (a-b) + (b-c)$$

$$\Rightarrow m \mid a-c$$

$$\Rightarrow a \equiv c \pmod{m}$$

$\Rightarrow \equiv$ is transitive.

Equivalence Class.

Suppose R is an equivalence relation on a non empty S . For each $a \in S$ let $[a]$

(8)

denote the set of elements of S to which a is related under R .

$$[a] = \{x \in S : xRa, (x, a) \in R\}.$$

$[a]$ is called equivalence class of a in S under R .

For example,

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$$

Let $m=4$.

$$[0] = \{0, \pm 4, \pm 8, \pm 12, \pm 16, \dots\}$$

$$[1] = \{\pm 1, \pm 5, \pm 9, \pm 13, \pm 17, \dots\}$$

$$[2] = \{\pm 2, \pm 6, \pm 10, \pm 14, \pm 18, \dots\}$$

$$[3] = \{\pm 3, \pm 7, \pm 11, \pm 15, \pm 19, \dots\}$$

Quotient Set.

The collection of all equivalence classes is denoted by S/R such that

$$S/R = \{[a] : a \in S\}$$

and R is called quotient set.

The quotient set of $\mathbb{Z}$ (for $m=4$) under Congruence relation is

$$\frac{\mathbb{Z}}{R} = \{[0], [1], [2], [3]\}.$$

In general,

$$\frac{\mathbb{Z}}{R_m} = \{[0], [1], [2], \dots, [m-1]\}$$

$$\underline{0}_m = \{0, 1, 2, \dots, m-1\}.$$

(9)

Cross Partition.

Let $\{A_1, A_2, A_3, \dots, A_m\}$ and $\{B_1, B_2, B_3, \dots, B_n\}$ be two partitions of a set X . Then the collection of sets $P = (\{A_i \cap B_j\}) \setminus \emptyset$ is also a partition of S and is known as cross partition.

Proof.

Let x be any element of a set X .

Since $\{A_1, A_2, \dots, A_m\}$ and $\{B_1, B_2, \dots, B_n\}$ are partitions of X so there exist i and j such that $x \in A_i$ and $x \in B_j$.

$$\Rightarrow x \in A_i \cap B_j$$

$\Rightarrow$ Union of $A_i \cap B_j$ is again X .

Now, suppose $A_k \cap B_s$ and $A_{k'} \cap B_{s'}$ are not disjoint.

Hence, there exist

$$x \in A_k \cap B_s \Rightarrow x \in A_k \text{ and } x \in B_s$$

$$\text{And } x \in A_{k'} \cap B_{s'} \Rightarrow x \in A_{k'} \text{ and } x \in B_{s'}$$

$$\Rightarrow x \in A_k \cap A_{k'} \text{ and } x \in B_s \cap B_{s'}$$

As $\{A_1, A_2, \dots, A_m\}$ and $\{B_1, B_2, \dots, B_n\}$ are partitions so only partition is

$$A_k = A_{k'} \text{ and } B_s = B_{s'}$$

Accordingly, $A_k \cap B_s = A_{k'} \cap B_{s'}$

$\Rightarrow$ Either the cells are disjoint or equal. Hence, P is a partition of X .

(10)

for example,

$$X = \{a, b, c, d\}$$

$$P_1 = \{\{a, b\}, \{c, d\}\}, \quad P_2 = \{\{a\}, \{b, c, d\}\}$$

$$P_3 = P_1 \cap P_2 = \{\{a\}, \{b\}, \{c, d\}\}.$$

Theorem.

Let R be an equivalence relation on a set S . Then the quotient set S/R is a partition of S .

Proof.

Let R be an equivalence relation on a set S . We have to show

- (i) for each $a \in S$, we have $a \in [a]$.
- (ii) $[a] = [b] \Leftrightarrow a R b$.
- (iii) $[a] \neq [b]$, then $[a]$ and $[b]$ are disjoint.

(i) For this, let $a \in S$. Since R is a reflexive relation, i.e. $(a, a) \in R$.

$$\text{Also } [a] = \{x \in S : (x, a) \in R\}.$$

$$\Rightarrow a \in [a].$$

(ii) Suppose $a R b$, i.e. $(a, b) \in R$.

To show $[a] = [b]$, let $x \in [a]$ then $(x, a) \in R$.

Also $(a, b) \in R$.

Since R is transitive so $(x, b) \in R$.

$$\Rightarrow x \in [b]$$

$$\Rightarrow [a] \subseteq [b].$$

(11)

Similarly we can show by the same argument,

$$[b] \subseteq [a]$$

Hence $[a] = [b]$.

Conversely. Suppose that $[a] = [b]$.

By (i) $b \in [b]$

$$\Rightarrow b \in [a] \quad \because [a] = [b].$$

$$\Rightarrow (a, b) \in R.$$

$$\Rightarrow a R b$$

(ii) Suppose on contrary that $[a]$ and $[b]$ are not disjoint.

$$\text{i.e. } [a] \cap [b] \neq \emptyset$$

Then there exists $x \in S$ such that
 $x \in [a]$ and $x \in [b]$.

$$\Rightarrow (a, x) \in R \text{ and } (x, b) \in R.$$

$$\Rightarrow (a, b) \in R \quad \because R \text{ is transitive.}$$

By (ii), $[a] = [b]$ which contradicts the assumption.

Hence the set of equivalence class defines partition.

Example.

Let $A = \{1, 2, 3, 4, 5, 6\}$ and

$$R = \{(1, 1), (1, 5), (2, 2), (2, 3), (2, 6), (3, 2), (3, 3), (3, 6), (4, 4), (5, 1), (5, 5), (6, 2), (6, 3), (6, 6)\}$$

be an equivalence relation.

(12)

Then find partition of A .

Solution.

$$[1] = \{x \in A : x R 1\}$$

(Definition)

$$= \{x \in A : (1, x) \in R\}$$

$$\Rightarrow [1] = \{1, 5\}$$

$$[2] = \{2, 3, 6\}$$

$$[4] = \{4\}$$

So partition is

$$P = \{\{1, 5\}, \{2, 3, 6\}, \{4\}\}$$

Theorem.

Suppose $P = \{A_i\}$ is a partition of a set

Then there exists an equivalence relation

$\sim$ on S such that $S/\sim$ is same as partition $P = \{A_i\}$.

Proof.

For $a, b \in S$, define $a \sim b$ if a and b belong to the same cell A_k in P .

We need to show that $\sim$ is an equivalence relation on S .

(i) Reflexive.

Let $a \in S$, by definition of partition there exists a cell A_k such that $a \in A_k$.

Hence, $a \sim a$.

Thus $\sim$ is reflexive.

(13)

(ii) Symmetric.

Suppose $a \sim b$. This implies that a and b belong to the same cell A_k .

$$\Rightarrow a, b \in A_k$$

$$\Rightarrow b, a \in A_k$$

سپیش میں ترتیب نہیں ہوتی

$$\Rightarrow b \sim a.$$

Thus $\sim$ is Symmetric.

(iii) Transitive.

Suppose $a \sim b$ and $b \sim c$. Then $a, b \in A_i$ and $b, c \in A_j$. Therefore, $b \in A_i \cap A_j$.

Since P is a partition, $A_i = A_j$.

Thus $a, c \in A_i$ and so $a \sim c$.

Thus $\sim$ is transitive.

Hence, $\sim$ is an equivalence relation on S .

Furthermore,

$$[a] = \{x : a \sim x\}$$

$$= \{x : x \text{ is in the same cell } A_k \text{ as } a\}.$$

Thus the equivalence classes under $\sim$ are the same as the cells in the partition P .

Example. :-

Let $A = \{1, 2, 3, 4, 5, 6\}$ and

$P = \{\{1\}, \{2, 3\}, \{4, 5, 6\}\}$ be a partition of A .

Find equivalence relation.

(14)

Solution.

The equivalence relation is given by
 $R = \{ (1,1), (2,2), (2,3), (3,2), (3,3), (4,4), (4,5), (5,4), (4,6), (6,4), (5,6), (6,5), (5,5), (6,6) \}$.

Example.

Let A be a set of non zero integers and let $\approx$ be the relation on $A \times A$ defined

$(a,b) \approx (c,d)$ whenever $ad = bc$.

(Q) Prove that $\approx$ is an equivalence relation

(i) Reflexive.

Let $(a,b) \in A \times A$. Since we know that integers are commutative so for $a,b \in A$.

$$ab = ba.$$

$$\Rightarrow (a,b) \approx (a,b).$$

Thus $\approx$ is reflexive.

(ii) Symmetric.

Suppose $(a,b) \approx (c,d)$.

$$\text{Then } ad = bc$$

$$\Rightarrow bc = ad$$

$$\Rightarrow cb = da$$

$\because$ commutative

$$\Rightarrow (c,d) \approx (a,b).$$

Thus $\approx$ is symmetric.

(iii) Transitive.

Suppose $(a,b) \approx (c,d)$ and $(c,d) \approx (e,f)$.

(15)

Then $ad = bc$ and $cf = de$

$$\Rightarrow \frac{a}{b} = \frac{c}{d} \quad \text{and} \quad \frac{c}{d} = \frac{e}{f}$$

$$\Rightarrow \frac{a}{b} = \frac{c}{d} = \frac{e}{f}$$

$$\Rightarrow \frac{a}{b} = \frac{e}{f} \Rightarrow af = be$$

$$\Rightarrow (a, b) \approx (e, f)$$

Thus $\approx$ is transitive.

Hence $\approx$ is an equivalence relation.

b) Find the equivalence class of $(2, 3) \in A \times A$ under $\approx$.

$$[(2, 3)] = \{(a, b) \in A \times A \text{ such that } (2, 3) \approx (a, b)\}$$

$$= \{(a, b) \in A \times A \text{ such that } 2b = 3a\}$$

$$= \{(a, b) \in A \times A \text{ such that } \frac{2}{3} = \frac{a}{b}\}$$

$$= \{(2, 3), (4, 6), (6, 9), (8, 12), \dots\}$$

Partial Ordering Relation.

A relation R on a set S is called ~~is called~~ a partial ordering of S or partial order on S if it has the following three properties:

① Reflexive i.e. $aRa \quad \forall a \in S$.

② Antisymmetric i.e. if aRb and bRa then $a = b$.

③ Transitive i.e. if aRb and bRc then aRc .

A set S together with a partial ordering R is called a partially ordered set or poset.

Examples:

- ① The relation $\leq$ on the set $\mathbb{R}$ of real numbers is partially ordering. ~~of any collection~~
- ② The relation $\subseteq$ of set inclusion is a partial ordering of any collection of sets.
- ③ The relation $a|b$ on the set of +ive integers is partial ordering.
- ④ Consider the set $\mathbb{Z}$ of integers define $a \sim b$ $b = a^{\pm 1}$ for some +ive integer ± 1 . Then $\sim$ is partial ordering.

(i) Reflexive.

Since $a = a^1$ so $a \sim a$. Thus $\sim$ is reflexive.

(ii) Antisymmetric.

Suppose $a \sim b$ and $b \sim a$ then there exist ± 1 and s such that

$$b = a^{\pm 1} \text{ --- (i) and } a = b^s \text{ --- (ii)}$$

$$(ii) \Rightarrow a^{\pm 1} = b^{s \pm 1}$$

$$(i) \Rightarrow b = a^{\pm 1} b^{s \pm 1}$$

There are four possibilities

(i) $s \pm 1 = 1$. Then $\pm 1 = 1$ and $s = 1$.

Since s and ± 1 are +ive integers so s and ± 1 should be 1.

This shows that $a = b$.

(ii) $a = 1$. Then $b = a^{\pm 1} = 1^{\pm 1} = 1 \Rightarrow b = 1$

$$\Rightarrow a = b$$

(17)

$$b=1 \cdot \text{Then } a=b^1=1^1=1 \Rightarrow a=1 \\ \Rightarrow a=b.$$

$$b=-1 \cdot \text{Then } a=b^1=(-1)^1=-1 \text{ or } 1$$

If $a=1$ then (by $b=a^1$) $b=1$ which is not possible.

$$\text{Hence } a=-1 \Rightarrow a=b.$$

Thus $\sim$ is antisymmetric.

transitive.

Suppose $a \sim b$ and $b \sim c$, there exist +ive integers and s such that $b=a^1$ and $c=b^s$ — (I)

$$\Rightarrow b^s = a^{1s} \text{ — (II)}$$

$$\text{By (I) and (II), } b^s = a^{1s} = c$$

$$\Rightarrow c = a^{1s} = a^t \text{ where } t=1s.$$

$$\Rightarrow c = a^t$$

$$\Rightarrow a \sim c$$

Thus $\sim$ is transitive.

Hence, $\sim$ is a partial ordering of $\mathbb{Z}$.

Equipotent sets.

Two sets A and B are said to be equipotent (or have the same cardinality), written as

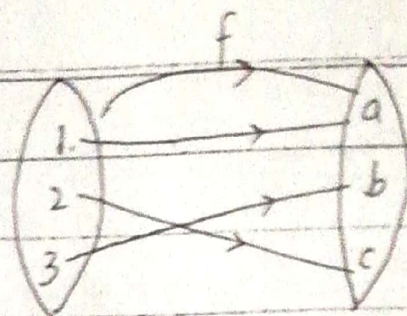
$A \approx B$ if there is a bijective function

$$f: A \rightarrow B.$$

Example.

$$A = \{1, 2, 3\} \text{ and } B = \{a, b, c\}.$$

(18)



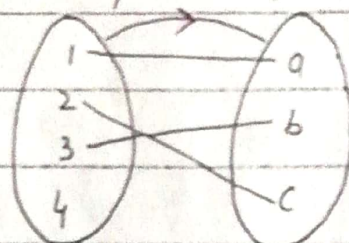
$$f(1) = a ; f(2) = c ; f(3) = b$$

f is one-one and onto.

Thus $A \approx B$.

Where as if $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$

Then



Here $A \not\approx B$.

Theorem: A/2011

The relation $A \approx B$ in sets is an equivalence relation.

Proof.

Reflexive. The identity function $I_A : A \rightarrow A$ is bijective, hence $A \approx A$.

Symmetric. Suppose $A \approx B$. Then there exists a bijective function $f : A \rightarrow B$. Hence f has an inverse function $f^{-1} : B \rightarrow A$ which is also bijective. Hence $B \approx A$.

Transitive. Suppose $A \approx B$ and $B \approx C$. Then there exist bijective functions $f : A \rightarrow B$ and $g : B \rightarrow C$. Then the composition function $g \circ f : A \rightarrow C$ is also bijective.

Hence $A \approx C$.

(19)

Example 1:

Show that $(0, 1) \approx (a, b)$.

Solution.

Define a function $f: (0, 1) \rightarrow (a, b)$ such that

$$f(x) = (b-a)x + a$$

 f is 1-1.Suppose $f(x_1) = f(x_2)$

$$\text{Then } (b-a)x_1 + a = (b-a)x_2 + a$$

$$\Rightarrow (b-a)x_1 = (b-a)x_2$$

$$\Rightarrow x_1 = x_2$$

 $\Rightarrow f$ is one-one. f is onto.For this, let $y \in (a, b)$.Now consider $x = \frac{y-a}{b-a}$

$$\begin{aligned} f(x) &= y \\ y &= (b-a)x + a \\ y-a &= (b-a)x \\ x &= \frac{y-a}{b-a} \end{aligned}$$

Since $a < y < b$.Therefore, $0 < y-a < b-a$

$$\Rightarrow \frac{0}{b-a} < \frac{y-a}{b-a} < \frac{b-a}{b-a}$$

$$\Rightarrow 0 < \frac{y-a}{b-a} < 1$$

$$\Rightarrow 0 < x < 1$$

$$\Rightarrow x \in (0, 1)$$

$$\text{Now, } f(x) = f\left(\frac{y-a}{b-a}\right)$$

(20)

$$= (b-a) \left(\frac{y-a}{b-a} \right) + a$$

$$= y - a + a = y$$

$$\Rightarrow f(x) = y$$

$\Rightarrow f$ is onto

Thus f is bijective.

Hence $(0,1) \approx (a,b)$.

Example 2:

Show that $(a,b) \approx (c,d)$.

Solution.

Define a function $f: (a,b) \rightarrow (c,d)$ such

$$\text{that } f(x) = \left(\frac{d-c}{b-a} \right)(x-a) + c$$

f is one-one.

$$\text{Suppose } f(x_1) = f(x_2)$$

$$\left(\frac{d-c}{b-a} \right)(x_1-a) + c = \left(\frac{d-c}{b-a} \right)(x_2-a) + c$$

$$\Rightarrow x_1 - a = x_2 - a$$

$$\Rightarrow x_1 = x_2$$

$\Rightarrow f$ is one-one.

f is onto.

For this, let $y \in (c,d)$.

$$\text{Now Consider } x = \left(\frac{b-a}{d-c} \right)(y-c) + a$$

Since $c < y < d$

Therefore, $0 < y-c < d-c$

(21)

$$\Rightarrow 0 < \frac{y-c}{d-c} < 1$$

$$\Rightarrow 0 < \frac{(y-c)}{(d-c)} (b-a) < b-a$$

$$\Rightarrow a < \frac{(b-a)}{d-c} (y-c) + a < b$$

$$\Rightarrow a < x < b$$

$$\Rightarrow x \in (a, b).$$

$$\begin{aligned} \text{Now } f(x) &= f\left(\frac{(b-a)}{d-c} (y-c) + a\right) \\ &= \frac{(d-c)}{b-a} \left(\frac{(b-a)}{(d-c)} (y-c) + a\right) + c \\ &= \left(y-c + \frac{(b-a)(d-c)}{b-a(d-c)}\right) + c \\ &= y-c+c = y \end{aligned}$$

$$\Rightarrow f(x) = y$$

$\Rightarrow f$ is onto.

Thus f is bijective function.

Hence $(a, b) \approx (c, d)$.

Example 3:

Show that $[0, 1] \approx (0, 1]$.

Solution.

Define a function $f: [0, 1] \rightarrow (0, 1]$ as

$$f(x) = 1-x$$

f is one-one.

$$\text{Suppose } f(x_1) = f(x_2).$$

$$\text{Then } 1-x_1 = 1-x_2$$

(22)

$$\Rightarrow x_1 = x_2$$

$$\Rightarrow f \text{ is } 1-1.$$

f is onto.

For this let $y \in (0, 1]$. Then $1-y \in [0, 1]$.

$$f(1-y) = y$$

$$\Rightarrow f \text{ is onto.}$$

$$\text{Hence } [0, 1] \approx (0, 1].$$

Infinite set.

A set S is said to be infinite if it has the same cardinality as a proper subset of itself. Otherwise S is finite.

$$\text{For example, } \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$$

$$E = \{0, \pm 2, \pm 4, \dots\}$$

$$\Rightarrow E \subset \mathbb{Z}$$

$f: \mathbb{Z} \rightarrow E$ is defined as

$$f(z) = 2z$$

clearly f is bijective.

$$\Rightarrow \mathbb{Z} \approx E \quad \text{although } E \subset \mathbb{Z}.$$

Example.

For any two sets A and B . Let $A' = A \times \{1\}$ and $B' = B \times \{2\}$. Then show that

$$A \approx A' \quad \text{and} \quad B \approx B'.$$

Solution.

$$\text{Clearly } A' \cap B' = \emptyset$$

(23)

(3.6)

prove $A \approx A'$ we define a mapping

$f: A \rightarrow A'$ such that

$$f(a) = (a, 1)$$

is one-one.

Suppose $f(a_1) = f(a_2)$

Then $(a_1, 1) = (a_2, 1)$

which is only possible if $a_1 = a_2$.

Hence f is 1-1.

f is onto.

(24)

Denumerable or Countably infinite sets.

A set D is said to be denumerable or Countably infinite if D has the same cardinality as P where P is the set of natural numbers.

that is $D \approx P$.

Countable sets. 2017,

A set is countable if it is finite or denumerable, and a set is non denumerable if it is not countable.

That is non denumerable sets are infinite and does not have the same cardinality as that of natural numbers, $P = \{1, 2, 3, \dots\}$.

Examples.

① The set of natural numbers is itself denumerable.

✓ ② The Cartesian Product $P \times P$ is denumerable
Solution.

② Define a mapping $f: P \times P \rightarrow P$ by
$$f(n, m) = 2^{n-1}(2m-1)$$

where $n, m \in P$.

f is one-one.

Suppose $f(n_1, m_1) = f(n_2, m_2)$

$$2^{n_1-1}(2m_1-1) = 2^{n_2-1}(2m_2-1)$$

$$12 = 2^2 \times 3$$

$$n = 2^5 \times \text{odd}$$

$$n = 2^{5-1}(2 \times 2 - 1)$$

representation
of natural
number

(25)

2 does not divide $(2m_1-1)$ and $(2m_2-1)$.

$$2^{n_1-1} = 2^{n_2-1}$$

$$\Rightarrow n_1 = n_2$$

$$2m_1-1 = 2m_2-1$$

$$\Rightarrow 2m_1 = 2m_2$$

$$\Rightarrow m_1 = m_2$$

$$\Rightarrow (m_1, m_1) = (m_2, m_2)$$

$\Rightarrow f$ is one-one.

f is onto.

$a \in P$. Since every natural number can be written as

$$a = 2^{n-1}(2m-1) \text{ where } n, m \in P.$$

we have $(n, m) \in P \times P$ such that

$$f(n, m) = 2^{n-1}(2m-1)$$

$\Rightarrow f$ is bijective.

$$\Rightarrow P \times P \approx P.$$

Let $N = \{0, 1, 2, 3, \dots\}$. Then show that

$N \times N$ is denumerable.

Proof.

Let $f: P \rightarrow N \times N$ be a mapping defined

$$f(a) = (n, m)$$

f is one-one.

Let $a = (n_1, m_1)$ and $b = (n_2, m_2)$.

(26)

Consider $f(a) = f(b)$.

Then by definition of mapping defined above

$$2^{n_1} (2m_1 + 1) = 2^{n_2} (2m_2 + 1)$$

for (n_1, m_1) and $(n_2, m_2) \in \mathbb{N} \times \mathbb{N}$.

Since 2 does not divide $(2m_1 + 1)$ and $(2m_2 + 1)$. Therefore,

$$2^{n_1} = 2^{n_2}$$

$$\Rightarrow n_1 = n_2$$

$$\text{and } 2m_1 + 1 = 2m_2 + 1$$

$$\Rightarrow 2m_1 = 2m_2$$

$$\Rightarrow m_1 = m_2$$

$$\Rightarrow (n_1, m_1) = (n_2, m_2)$$

$$\Rightarrow a = b$$

f is onto.

Since every natural number 'a' can be written as

$$a = 2^n (2m + 1)$$

Then there exists $(n, m) \in \mathbb{N} \times \mathbb{N}$ such that

$$f(n, m) = 2^n (2m + 1) = a$$

$$\Rightarrow f \text{ is onto.}$$

Thus $P \approx \mathbb{N} \times \mathbb{N}$.

(27)

Theorem 2.14 :-

Every infinite set contains a subset which is denumerable.

Proof. :-

Let A be an infinite set and $P(A)$ be its power set. Let $f: P(A) \rightarrow A$ be a choice function such that

$$f(A) = a_1$$

$$f(A \setminus \{a_1\}) = a_2$$

$$f(A \setminus \{a_1, a_2\}) = a_3$$

$$\vdots$$

$$f(A \setminus \{a_1, a_2, \dots, a_{n-1}\}) = a_n.$$

Since A is infinite, $A \setminus \{a_1, a_2, \dots, a_{n-1}\}$ is not empty for every $n \in \mathbb{P}$ ($\mathbb{P}$ is set of natural numbers).

Also, ~~since~~ f is a choice function.

So $a_n \neq a_i$ for $i < n$.

In this way, we get a sequence $\{a_1, a_2, \dots, a_n, \dots\}$ which is denumerable subset of A .

Theorem.

A subset of a denumerable set is either finite or denumerable.

Proof.

Let $A = \{a_1, a_2, \dots, a_{n-1}, a_n, \dots\} \longrightarrow \textcircled{1}$
 be a denumerable set and B be its subset.

(28)

If $B = \emptyset$, then clearly B is finite.

Suppose that $B \neq \emptyset$.

Let b_1 be the first element in the sequence in ① such that $b_1 \in B$. Let b_2 be the first element which follows b_1 in the sequence in ① such that $b_2 \in B$, and so on.

Then $B = \{b_1, b_2, \dots\}$. If the sequence $b_1, b_2, \dots$ ends then B is finite, otherwise B is denumerable.

Theorem.

A subset of countable set is countable.

Proof.

Let A be a countable set. Then either A is finite or denumerable.

If A is finite, then any subset of A is also finite, so is countable.

Suppose A is denumerable and B is its subset. If $B = \emptyset$ then clearly B is finite. Suppose that $B \neq \emptyset$.

Let b_1 be the first element in the sequence in $A = \{a_1, a_2, \dots\}$ such that $b_1 \in B$.

Let b_2 be the first element which follows b_1 in the sequence in $A = \{a_1, a_2, \dots\}$ such that $b_2 \in B$, and so on.

Then $B = \{b_1, b_2, \dots\}$. If the sequence $b_1, b_2, \dots$

(29)

ends then B is finite, otherwise B is denumerable. Hence B is countable.

Theorem.

If A is denumerable, then $A \cup \{x\}$ is also denumerable.

Proof.

If $x \in A$, then $A \cup \{x\} = A$.

$\Rightarrow A \cup \{x\}$ is denumerable.

We assume $x \notin A$. Since A is denumerable

so there exists a bijective mapping $f: P \rightarrow A$, where $P = \{1, 2, 3, \dots\}$.

Define a mapping $g: P \rightarrow A \cup \{x\}$ such that

$$g(n) = \begin{cases} x & \text{if } n=1 \\ f(n-1) & ; \text{ otherwise} \end{cases}$$

Now we show that g is bijective.

g is 1-1.

For $n_1, n_2 \in P$, suppose that

$$g(n_1) = g(n_2)$$

Since $x \notin A$ and $g(1) = x$ so $n_1, n_2 \neq 1$.

$$\Rightarrow f(n_1 - 1) = f(n_2 - 1)$$

Since f is bijective.

$$\text{Therefore, } n_1 - 1 = n_2 - 1$$

$$\Rightarrow n_1 = n_2$$

$$\Rightarrow g \text{ is 1-1.}$$

(30)

g is onto.

For this, let $y \in A \cup \{x\}$.

If $y = x$ then $g(1) = x = y$

If $y \neq x$ then $y \in A$.

Since $f: P \rightarrow A$ is onto, there exists $n \in P$ such that

$$y = f(n).$$

$$\Rightarrow y = f(n+1-1) = f(n-1+1)$$

$$\Rightarrow y = g(n+1), \quad n+1 \in P.$$

$\Rightarrow g$ is onto.

$\Rightarrow g$ is bijective.

Theorem.

If A is finite and B is denumerable. Then $A \cup B$ is denumerable.

Proof.

Suppose $A = \{a_1, a_2, \dots, a_k\}$. Since B is denumerable so there exists a bijective mapping $f: P \rightarrow B$.

Consider $A' = A \setminus B = \{a'_1, a'_2, \dots, a'_s\}$.

Clearly $A \cup B = A' \cup B$.

To prove $A' \cup B$ is denumerable, we define a mapping $g: P \rightarrow A' \cup B$ such that

$$g(n) = \begin{cases} a'_n & \text{if } n \leq s \\ f(n-s) & \text{otherwise} \end{cases}$$

(31)

g is one-one.

For $n_1, n_2 \in P$.

Suppose $g(n_1) = g(n_2)$

Since $A' \cap B = \emptyset$ so either $n_1, n_2 \leq s$
or $n_1, n_2 > s$.

Suppose $n_1, n_2 \leq s$

$$\Rightarrow a'_{n_1} = a'_{n_2}$$

Since a'_i are distinct so $n_1 = n_2$

If $n_1, n_2 > s$, then

$$f(n_1 - s) = f(n_2 - s)$$

Since f is bijective, f is one-one.

$$\text{i.e. } n_1 - s = n_2 - s$$

$$\Rightarrow n_1 = n_2$$

$\Rightarrow g$ is one-one.

g is onto:

For this let $y \in A' \cup B$, then either $y \in A'$
or $y \in B$.

If $y \in A'$. Then $y = a'_i$ for some $i \in P : i \leq s$

As $i \leq s$, so $g(i) = a'_i = y$

Now suppose $y \in B$. Since $f: P \rightarrow B$ is
onto, there exists $n \in P$ such that

$$y = f(n).$$

Since n and s are natural numbers,

$n+s$ also a natural number,

$$\text{i.e. } n+s \in P.$$

(32)

$$g(n+s) = f(n+s+s) \\ = f(n)$$

$$\Rightarrow g(n+s) = y$$

$\Rightarrow g$ is onto.

Thus g is bijective which implies that $A' \cup B$ is denumerable.

Since $A' \cup B = A \cup B$.

Therefore, $A \cup B$ is denumerable.

Theorem. 2011, 2016,

The set $\mathbb{Q}$ of rational numbers is denumerable.

Proof.

Since $\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$ where $\mathbb{Q}^+$ and $\mathbb{Q}^-$ denote respectively the sets of +ive and -ive rational numbers.

Let $f: \mathbb{Q}^+ \rightarrow \mathbb{P} \times \mathbb{P}$ be defined by

$$f(p/q) = (p, q). \text{ for } p/q \in \mathbb{Q}^+$$

Where p/q is any element of $\mathbb{Q}^+$ expressed as the ratio of two relatively prime +ive integers.

Then f is one to one and so $\mathbb{Q}^+$ has the same cardinality as a subset of $\mathbb{P} \times \mathbb{P}$.

Since the Cartesian product is denumerable

$\mathbb{P} \times \mathbb{P}$ is denumerable.

Since a subset of denumerable set is

(33)

or denumerable, the set Q^+ is denumerable.
Similarly Q^- is denumerable.

Thus the set Q of rational numbers, the union of Q^+ , $\{0\}$ and Q^- , is also denumerable.

Characteristic function.

Let U be a universal set and $A \subset U$ then characteristic function χ_A is defined as

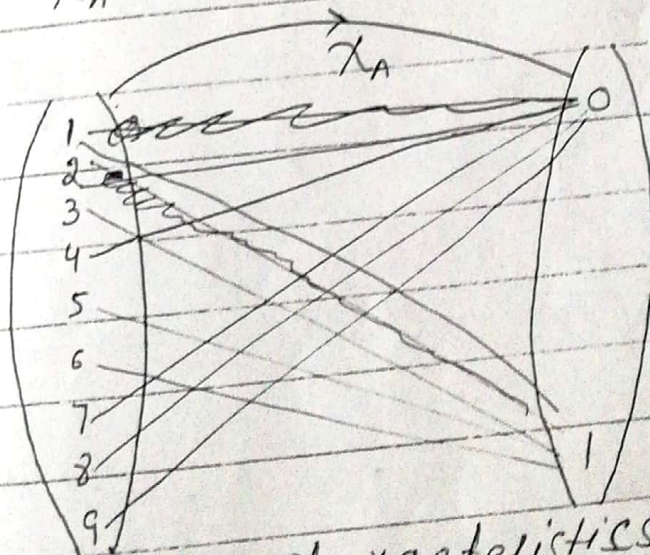
$$\begin{aligned} \textcircled{*} \chi_A : U &\rightarrow \{0, 1\} \\ \chi_A(x) &= \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases} \end{aligned}$$

For example.

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$A = \{1, 3, 5, 6\}$$

Then $\chi_A : U \rightarrow \{0, 1\}$ is defined as



represents a characteristics function.

Theorem.

Let X be any set and let $C(X)$ be the family of characteristic functions of X .

(34)

Then $P(X) \approx C(X)$ where $P(X)$ is the power set of X .

Proof.

Let A be any subset of X i.e. $A \in P(X)$

Let $f: P(X) \rightarrow C(X)$ be a function defined as $f(A) = \chi_A$.

f is one-one.

Let $A, B \in P(X)$ and suppose that

$$f(A) = f(B)$$

$$\text{Then } \chi_A = \chi_B$$

$$\Rightarrow A = B$$

$\Rightarrow f$ is one-one.

f is onto.

For this, let $\chi_A \in C(X)$ then there exists an $A \in P(X)$ such that

$$f(A) = \chi_A$$

$\Rightarrow f$ is onto.

Thus f is bijective.

Hence $P(X) \approx C(X)$.

Theorem. 2016,

Let A and B be two denumerable. then $A \cup B$ is also denumerable.

Proof.

Let $S_1 = A$ and $S_2 = B \setminus A$.

Then $S_1 \cup S_2 = A \cup B$

(35)

and $S_1 \cap S_2 = \emptyset$.

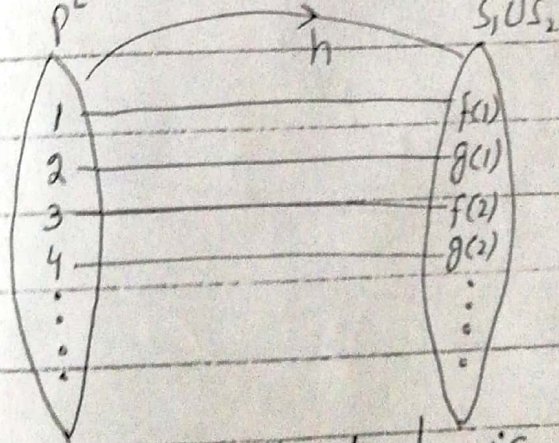
If S_2 is finite then we know that union of finite and denumerable set is also denumerable, so $S_1 \cup S_2$ is denumerable, and hence $A \cup B$ is denumerable.

Thus we may suppose S_2 is denumerable.

Since S_1 and S_2 are denumerable, there exist bijective mappings $f: P \rightarrow S_1$ and $g: P \rightarrow S_2$.

Consider a function $h: P \rightarrow S_1 \cup S_2$ as

$$h(n) = \begin{cases} f(\frac{n+1}{2}) & \text{if } n \text{ is odd} \\ g(\frac{n}{2}) & \text{if } n \text{ is even} \end{cases} \quad ; n \in P$$



Now we show that h is bijective.

h is one-one.

For $n_1, n_2 \in P$, suppose that

$$h(n_1) = h(n_2)$$

Since $S_1 \cap S_2 = \emptyset$ so ~~there~~ either n_1 and n_2 both are odd or both are even.

(36)

Case 1:

If n_1 and n_2 are odd then

$$h(n_1) = h(n_2)$$

$$\Rightarrow f\left(\frac{n_1+1}{2}\right) = f\left(\frac{n_2+1}{2}\right)$$

Since f is one-one, therefore,

$$\frac{n_1+1}{2} = \frac{n_2+1}{2}$$

$$\Rightarrow n_1+1 = n_2+1$$

$$\Rightarrow n_1 = n_2$$

Case 2:

If n_1 and n_2 are even then

$$h(n_1) = h(n_2)$$

$$\Rightarrow g\left(\frac{n_1}{2}\right) = g\left(\frac{n_2}{2}\right)$$

Since g is one-one, therefore,

$$\frac{n_1}{2} = \frac{n_2}{2}$$

$$\Rightarrow n_1 = n_2$$

In either case h is 1-1. h is onto.For this, let $y \in S_1 \cup S_2$, then $y \in S_1$ or $y \in S_2$. Suppose $y \in S_1$. As f is onto so there exists $n_1 \in P$ such that

$$y = f(n_1)$$

(37)

Let $n = 2n_1 - 1$, then clearly n is odd.

$$\begin{aligned} \frac{n+1}{2} &= n_1 \\ n &= 2n_1 - 1 \end{aligned}$$

$$\begin{aligned} h(n) &= f\left(\frac{n+1}{2}\right) \\ &= f\left(\frac{2n_1 - 1 + 1}{2}\right) = f(n_1) = y \end{aligned}$$

$$\Rightarrow h(n) = y.$$

Now suppose $y \in S_2$. As g is onto so there exists $n_2 \in P$ such that

$$y = g(n_2)$$

Let $n = 2n_2$, then clearly n is even.

$$\begin{aligned} \frac{n}{2} &= n_2 \\ n &= 2n_2 \end{aligned}$$

$$\begin{aligned} h(n) &= g\left(\frac{n}{2}\right) \\ &= g\left(\frac{2n_2}{2}\right) = g(n_2) = y \end{aligned}$$

$$\Rightarrow h(n) = y$$

$\Rightarrow$ In either case h is onto.

$\Rightarrow h$ is bijective function.

Hence $A \cup B$ is denumerable.

Theorem 1.2017, A/2013,

Countable union of countable sets is countable.

Proof.

Let $C = \{S_i : i \in P\}$ be collection of countable sets.

$$\text{Let } A_1 = S_1$$

$$A_2 = S_2 \setminus S_1$$

(38)

$$A_3 = S_3 \setminus S_1 \cup S_2$$

$$A_4 = S_4 \setminus S_1 \cup S_2 \cup S_3$$

and so on.

Then clearly $\bigcup_{i \in P} A_i = \bigcup_{i \in P} S_i$ and $C^* = \{A_i : i \in P\}$ is the collection of pairwise disjoint countable sets.

Now there are three cases.

- ① Suppose all A_i 's are finite.
- ② Suppose all A_i 's are denumerable.
- ③ Some A_i 's are finite and others are denumerable.

Case 1:

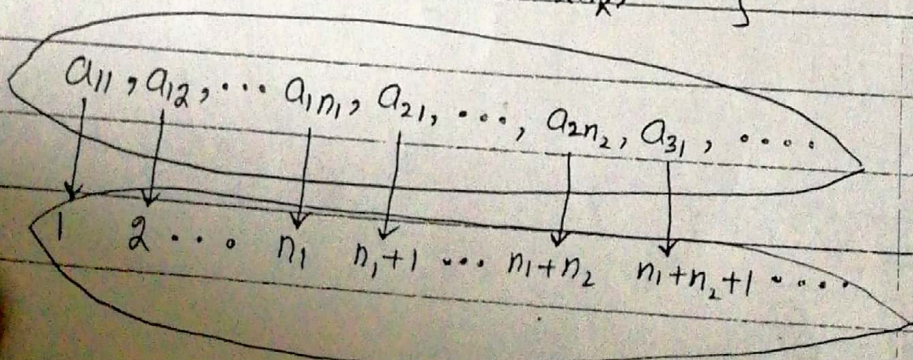
$$\text{Let } A_1 = \{a_{11}, a_{12}, \dots, a_{1n_1}\}$$

$$A_2 = \{a_{21}, a_{22}, \dots, a_{2n_2}\}$$

$$\vdots$$

$$A_k = \{a_{k1}, a_{k2}, \dots, a_{kn_k}\}$$

$$\text{Then } \bigcup_{i \in P} A_i = \{a_{11}, a_{12}, \dots, a_{1n_1}, a_{21}, a_{22}, \dots, a_{2n_2}, a_{k1}, a_{k2}, \dots, a_{kn_k}, \dots\}$$



(39)

clearly from fig: $n_1 + n_2 + \dots + n_{i-1} + j$.

Now we define a function $f: \bigcup_{i \in P} A_i \rightarrow P$ such that

$$f(a_{ij}) = n_1 + n_2 + \dots + n_{i-1} + j$$

f is one-one.

For $a_{ij}, a_{i'j'} \in \bigcup_{i \in P} A_i$, suppose that

$$f(a_{ij}) = f(a_{i'j'})$$

we have to show $i = i'$ and $j = j'$.

Suppose on contrary $i \neq i'$ and without any loss of generality we assume that $i > i'$.

Now, $f(a_{ij}) = f(a_{i'j'}) : (1 \leq j \leq n_i \text{ and } 1 \leq j' \leq n_{i'})$

$$\Rightarrow n_1 + n_2 + \dots + n_{i-1} + j = n_1 + n_2 + \dots + n_{i'-1} + j' \quad \text{--- (1)}$$

As $i > i'$ so we can write (1) سے جو مساوی ہے اس میں 'i' کی جگہ 'i'' لکھیں اور 'j' کی جگہ 'j'' لکھیں

$$n_1 + n_2 + \dots + n_{i'-1} + n_{i'} + \dots + n_{i-1} + j = n_1 + n_2 + \dots + n_{i'-1} + j'$$

$$\Rightarrow n_{i'} + \dots + n_{i-1} = j' - j > 0 \quad (\text{as } \sum n_j > 0)$$

$$\text{As } j' \leq n_{i'} \text{ so } j' - j \leq n_{i'}$$

$$\Rightarrow n_{i'} + \dots + n_{i-1} \leq n_{i'}$$

which is not possible as each n_i is +ive.

Thus $i = i'$.

Substituting $i = i'$ in (1), we get

$$j = j'$$

$$\Rightarrow a_{ij} = a_{i'j'}$$

$\Rightarrow f$ is one-one.

(40)

f is onto.

For this, let $y \in P$. Choose the largest i such that

$$y - (n_1 + n_2 + \dots + n_{i-1}) > 0$$

and consider

$$j = y - (n_1 + n_2 + \dots + n_{i-1})$$

with the choice of i and j ,

$$a_{ij} \in \bigcup_{i \in P} A_i$$

$$\begin{aligned} \text{and } f(a_{ij}) &= n_1 + n_2 + \dots + n_{i-1} + j \\ &= (n_1 + n_2 + \dots + n_{i-1}) + y - (n_1 + n_2 + \dots + n_{i-1}) \end{aligned}$$

$$\Rightarrow f(a_{ij}) = y$$

$$\Rightarrow f \text{ is onto.}$$

Thus f is bijective function.

$\Rightarrow$ countable union of finite sets is countable

Case 2:

$$\text{Suppose } A_1 = \{a_{11}, a_{12}, \dots\}$$

$$A_2 = \{a_{21}, a_{22}, \dots\}$$

$$\vdots$$

$$A_n = \{a_{n1}, a_{n2}, \dots\}$$

$$\vdots$$

$$\text{Define } D_n = \{a_{ij} : i+j=n, n>1\}$$

$$\text{Then } D_2 = \{a_{11}\}, D_3 = \{a_{12}, a_{21}\}$$

$$D_4 = \{a_{13}, a_{31}, a_{22}\}, \dots$$

(41)

Clearly all D_n 's are finite and $\bigcup_{i \in I} A_i = \bigcup_{i \in I} D_i$.
Since we know that Countable union of finite sets is countable so $\bigcup_{i \in I} D_i$ is countable.

Hence $\bigcup_{i \in I} A_i$ is countable.

Case 3:

Let $C_1 \subseteq C^*$ be the collection of all finite sets in C^* and $C_2 \subseteq C^*$ be the collection of all denumerable sets in C^* .

Then clearly $C^* = C_1 \cup C_2$

By Case 1, C_1 is countable (finite or denumerable) and by Case 2, C_2 is countable (denumerable).

In either case C^* is countable and hence C is countable.

Hence Countable union of Countable sets is Countable.

Theorem.

Suppose A is an infinite set and F is a finite subset of A . Show that $A \setminus F \approx A$.

Removing a finite number of elements from an infinite set does not change its cardinality.

Proof.

Suppose $F = \{a_1, a_2, \dots, a_n\}$. Since every infinite set has denumerable subset, we choose a denumerable subset $D = \{a_1, a_2, \dots, a_n, a_{n+1}, \dots\}$

(42)

of A so that the first n elements of D are the elements of F .

Let $g: A \rightarrow A/F$ be a mapping defined by

$$g(a) = a, \text{ if } a \notin D, \quad a \in A$$

and $g(a_k) = a_{k+n}$ if $a \in D$.

g is 1-1.

If $a_1, a_2 \notin D$ then $g(a_1) = g(a_2)$

$$\Rightarrow a_1 = a_2$$

If $a_1, a_2 \in D$ then $g(a_{k_1}) = g(a_{k_2})$

$$\Rightarrow a_{k_1+n} = a_{k_2+n}$$

$$\Rightarrow k_1+n = k_2+n$$

$$\Rightarrow k_1 = k_2$$

So there exists a bijective mapping $f: D \rightarrow$

$$\text{as } f(n) = a_n$$

$$f(a_1) = f(a_2)$$

Since f is one-one,

$$\text{so } a_1 = a_2$$

f is onto.

For this either $a \in D$ or $a \notin D$.

If $a \notin D$, then there exists $a \in A/F$ such that

$$g(a) = a$$

If $a \in D$, $a = a_i$ for some $i \geq n$.

Then there exist a_{i-1} such that

$$g(a_{i-1}) = a_i$$

Thus g is bijective. Hence $A/F \cong A$.

(43)

2018, 2014, Apr 2018 (uncountable), Apr 2012,

the closed interval $[0, 1]$ is non denumerable.

Proof.

Suppose $I = [0, 1]$ is denumerable. Then $I = [0, 1]$ can be written as

$$I = \{x_1, x_2, x_3, \dots\}$$

i.e. the elements of I can be written in a sequence. Now we construct a sequence of closed intervals. Consider the following three closed subintervals of $I = [0, 1]$.

$$[0, \frac{1}{3}], [\frac{1}{3}, \frac{2}{3}], [\frac{2}{3}, 1] \longrightarrow \textcircled{1} \quad \begin{array}{c} | \quad | \quad | \quad | \\ 0 \quad \frac{1}{3} \quad \frac{2}{3} \quad 1 \end{array}$$

where each interval has equal length $\frac{1}{3}$.

Now, x_1 cannot belong to all three intervals.

If x_1 is one of the endpoints, then it could belong to two of the intervals but not all three.

Let $I_1 = [a_1, b_1]$ be the closed sub-interval of $[0, 1]$

such that $x_1 \notin [a_1, b_1] = I_1$

Now consider the following three closed interval of $[a_1, b_1] = I_1$,

$$[a_1, a_1 + \frac{1}{9}], [a_1 + \frac{1}{9}, a_1 + \frac{2}{9}], [a_1 + \frac{2}{9}, a_1 + \frac{3}{9} = b_1]$$

with each has length $\frac{1}{9}$.

Similarly, let $I_2 = [a_2, b_2]$ be the closed sub-interval of $[a_1, b_1]$ such that

(44)

$$x_2 \notin [a_2, b_2] = I_2$$

In this way we get a sequence in such a way $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ such that $x_i \notin I_i, i \in \mathbb{P}$.

Since nested sequence of closed intervals has a non empty intersection, there exists a real number y such that

$$y \in \bigcap_{i \in \mathbb{P}} I_i \neq \emptyset.$$

Then $y \in I_i \quad \forall i \quad \longrightarrow \textcircled{2}$

$$\text{Also } y \in I = \{x_1, x_2, \dots\}$$

$\Rightarrow y = x_j$ for some $j \in \mathbb{P}$.
By our construction we must have $y = x_j \notin I_j \longrightarrow \textcircled{3}$

$\textcircled{2}$ and $\textcircled{3}$ contradicts each other.

Hence $[0, 1]$ is non-denumerable.

Power of Continuum.

A set A is said to have the power of Continuum if it has the same cardinality of the unit interval $[0, 1]$.

Remarks:

① All intervals $([a, b], (a, b), [a, b), (a, b])$ have the power of Continuum.

② The set of real numbers has power of Continuum.
Proof.

Define a function $f: \mathbb{R} \rightarrow D = (-1, 1)$ as

$$f(x) = \frac{x}{1 + |x|}$$

$$N_0 = \mathbb{C}$$

$$\mathbb{C}^{N_0} = \mathbb{C}$$

(45)

Then clearly $f(x) \in D = (-1, 1)$ as $|x| \leq 1 + |x|$.

Now we show that f is bijective.

f is 1-1.

For $x_1, x_2 \in \mathbb{R}$ suppose that $f(x_1) = f(x_2)$.

Then $f(x_1)$ and $f(x_2)$ have same sign. As

$1 + |x| > 0 \quad \forall x$ so $f(x)$ and x have same

sign and hence x_1 and x_2 have same sign.

Now
$$f(x_1) = f(x_2)$$

$$\Rightarrow \frac{x_1}{1 + |x_1|} = \frac{x_2}{1 + |x_2|}$$

$$\Rightarrow x_1(1 + |x_2|) = x_2(1 + |x_1|)$$

$$\Rightarrow x_1 + x_1|x_2| = x_2 + x_2|x_1| \quad \text{--- (1)}$$

Case 1:

If x_1, x_2 are +ive real numbers then

$$x_1 + x_1x_2 = x_2 + x_2x_1$$

$$\Rightarrow x_1 = x_2$$

$$\Rightarrow f \text{ is } 1-1.$$

Case II:

If x_1, x_2 are -ive real numbers then

$$x_1 + x_1(-x_2) = x_2 + x_2(-x_1)$$

$$\Rightarrow x_1 - x_1x_2 = x_2 - x_2x_1$$

$$\Rightarrow x_1 = x_2 \Rightarrow f \text{ is one-one.}$$

f is onto.

Cardinal numbers.

A set A is assigned a symbol in such a way that two sets A and B are assigned the same symbol if and only if they are equipotent. This symbol is called Cardinality or Cardinal number of A .

It is denoted by $|A|$, $n(A)$, $\#(A)$ or $\text{Card}(A)$.

$$\text{i.e. } |A| = |B| \Leftrightarrow A \sim B.$$

Finite Cardinal numbers.

$0 \rightarrow$ empty set, $1 \rightarrow$ set containing one element

$\dots n \rightarrow$ set containing n elements.

$$A = \{1, 2, 3, \dots, n\}$$

$$|B| = n \text{ if } A \sim B.$$

Transfinite Cardinal number.

Cardinal numbers of infinite sets are called infinite or transfinite Cardinal numbers.

The notation for $\aleph_0$ Cardinal numbers is $\aleph_0$.

$\aleph_0$ = aleph nought which is first letter of the Hebrew alphabet.

$$|A| = \aleph_0 \Leftrightarrow A \sim \mathbb{N}$$

$$|A| = C \Leftrightarrow A \sim [0, 1]$$

where C is power of Continuum.

Ordering Relations in Cardinal numbers.

Let A and B be two sets. Then we say

$|A| \leq |B|$ if A has the same cardinality

(47)

a subset of B . ($A \sim$ subset of B).

equivalently there is one-one function $A \rightarrow B$.

notes.

$$|A| < |B| \Leftrightarrow |A| \leq |B|, |A| \neq |B|.$$

$$A \sim A', B \sim B'$$

$$|A| \leq |B| \Leftrightarrow |A'| \leq |B'|$$

Cantor's Theorem. 2012, 2014, 2015,

for any set A , we have $|A| < |P(A)|$.

for infinite set

Proof.

Let A be any set. Consider a function $f: A \rightarrow P(A)$ defined as

$$f(a) = \{a\} \in P(A) \quad \text{for } a \in A.$$

f is one-one.

$$\text{Let } f(a) = f(b)$$

$$\Rightarrow \{a\} = \{b\}$$

$$\Rightarrow a = b$$

$\Rightarrow f$ is one-one.

Hence by definition $|A| \leq |P(A)| \longrightarrow \textcircled{1}$

Now we show that $|A| \neq |P(A)|$.

Suppose on contrary $|A| = |P(A)|$. Then there exists a bijection $f: A \rightarrow P(A)$.

Consider a set $B = \{x \in A : x \notin f(x)\}$.

Then clearly $B \in P(A)$.

(48)
Since f is onto, there exists an element $a \in A$ such that

$$f(a) = B \longrightarrow (2)$$

Then either $a \in B$ or $a \notin B$.

If $a \in B$. Then $a \notin f(a) = B$ (from ②)
 $\Rightarrow a \notin B$, a contradiction.

If $a \notin B$. Then $a \in f(a) = B$ (from ②)
which is again a contradiction.

Thus original assumption that $|A| = |P(A)|$ is not possible.

Hence $|A| < |P(A)|$.

Note:

For any finite cardinal number n , $n < 2^n$.

V. Imp, Schroeder - Bernstein Theorem. 2013, 2015,

If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Proof.

To prove the required result, we will prove its equivalent form i.e. for the sets X, Y, X_1 such that $X \supseteq Y \supseteq X_1$ and $X \approx X_1$, then $X \approx Y$.

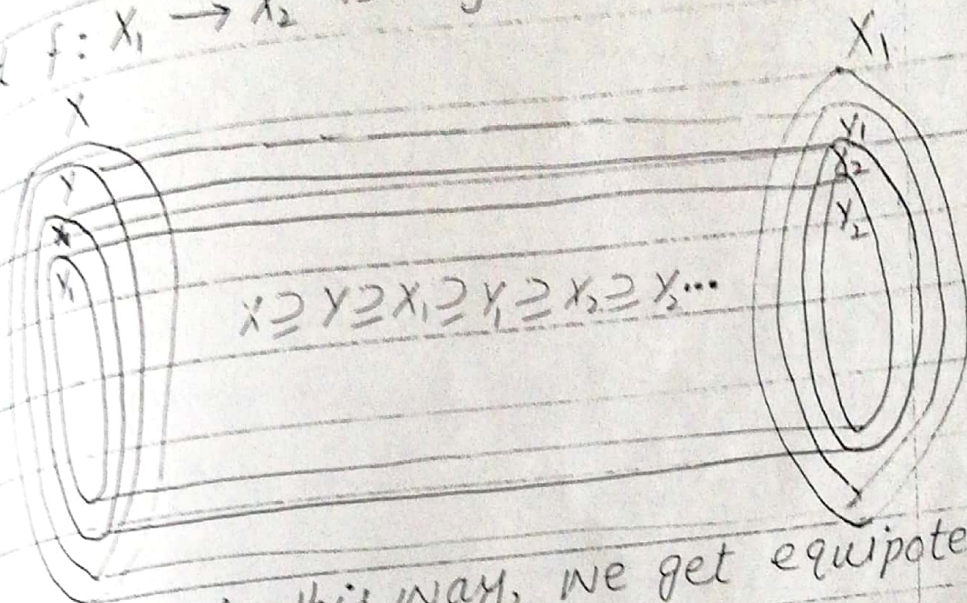
For this let $X \supseteq Y \supseteq X_1$ and $X \approx X_1$. Then there exists a bijection $f: X \rightarrow X_1$.

Since $Y \subseteq X$, the restriction of f on Y is also one-one, therefore, $f(Y) \approx Y_1 \subseteq X_1$.

$$\Rightarrow X \supseteq Y \supseteq X_1 \supseteq Y_1$$

(49)

$f: Y \rightarrow X_1$ is also bijection.
 Now consider $Y \supseteq X_1 \supseteq X_1$ and $Y \approx X_1$.
 $X_1 \subseteq Y$ so $f(X_1) \approx X_2 \subseteq Y$
 $\Rightarrow X \supseteq Y \supseteq X_1 \supseteq X_1 \supseteq X$
 and $f: X_1 \rightarrow X_2$ is bijection.



Continuing in this way, we get equipotent sets $X, X_1, X_2, \dots$ and equipotent sets $Y, Y_1, Y_2, \dots$ such that
 $X \supseteq Y \supseteq X_1 \supseteq Y_1 \supseteq X_2 \supseteq Y_2 \supseteq X_3 \supseteq \dots$ and
 $f: X_k \rightarrow X_{k+1}$ and $f: Y_k \rightarrow Y_{k+1}$ are bijections.

Let $B = X \cap Y \cap X_1 \cap Y_1 \cap X_2 \cap Y_2 \cap \dots$

Then $X = (X \setminus Y) \cup (Y \setminus X_1) \cup (X_1 \setminus Y_1) \cup (Y_1 \setminus X_2) \cup \dots \cup B$

$Y = (Y \setminus X_1) \cup (X_1 \setminus Y_1) \cup (Y_1 \setminus X_2) \cup (X_2 \setminus Y_2) \cup \dots \cup B$

where $X \setminus Y, X_1 \setminus Y_1, X_2 \setminus Y_2, \dots$ are equipotent.

Also the function $f: (X_k \setminus Y_k) \rightarrow (X_{k+1} \setminus Y_{k+1})$ is

clearly bijective.

Consider the function $g: X \rightarrow Y$ defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \in X_k \setminus Y_k \text{ or } x \in X \setminus Y \\ x & \text{if } x \in Y_k \setminus X_k \text{ or } x \in B. \end{cases}$$

(51)

Therefore, from ① we can see that,
 $q \in f(x_2)$ but $q \notin f(x_1)$

$$\Rightarrow f(x_1) \neq f(x_2)$$

$\Rightarrow f$ is one-one.

$$\Rightarrow |R| \leq |P(Q)|$$

$$\Rightarrow C \leq 2^{\aleph_0} \longrightarrow \textcircled{2}$$

Now, let $C(P)$ be the family of all characteristic functions $f: P \rightarrow \{0, 1\}$, where $P = \{1, 2, 3, \dots\}$.

We know that $C(P) \approx P(P)$.

Now define a mapping $\phi: C(P) \rightarrow [0, 1]$

such that $\phi(f) = 0.f(1)f(2)f(3)\dots$ an infinite decimal consisting of zero's or one's.

To prove ϕ is one-one, let $f, h \in C(P)$ such that $f \neq h$

So there exists $n \in P$ such that

$$f(n) \neq h(n).$$

Then the decimals would be different as

$$\phi(f) = 0.f(1)f(2)\dots f(n)$$

$$\phi(h) = 0.h(1)h(2)\dots h(n)$$

$$\Rightarrow \phi(f) \neq \phi(h)$$

$\Rightarrow \phi$ is one-one.

$$\Rightarrow |C(P)| \leq |[0, 1]|$$

$$\Rightarrow |C(P)| \leq C$$

$$\text{As } C(P) \approx P(P) \approx P(Q)$$

$$\text{So } |P(Q)| \leq C$$

$$\Rightarrow 2^{\aleph_0} \leq C \longrightarrow \textcircled{3}$$

hint

$$P(Q) \rightarrow \mathbb{R}$$

$$P(P) \rightarrow [0, 1]$$

$$C(P) \rightarrow [0, 1]$$

e.g.

$$X = \{1, 3\}$$

$$f(1) = 1$$

$$f(2) = 0$$

$$f(3) = 1$$

$$f(4) = 0$$

$$\dots$$

$$\Rightarrow \phi(f) = 0.1010\dots$$

(52)

From ② and ③, and using S. Bernstein theorem
i.e. if for cardinal numbers α, β , $\alpha \leq \beta$ and $\beta \leq \alpha$,
we get, $C = 2^{N_0}$.

Continuum hypothesis.

There exists no cardinal number β such that

$$N_0 < \beta < C.$$

Cardinal Arithmetic.

Cardinal Addition.

Let α and β be cardinal numbers and
let A and B be disjoint sets with $|A| = \alpha$
and $|B| = \beta$. Then the sum of α and β is
defined as

$$\alpha + \beta = |(A \cup B)|$$

Remarks.

① First note that this condition is well
defined. Suppose $|A| = \alpha = |A'|$ and
 $|B| = \beta = |B'|$ then

$$|(A' \cup B')| = \alpha + \beta$$

$$|(A \cup B)| = \alpha + \beta$$

$$\Rightarrow |(A' \cup B')| = |(A \cup B)|$$

② If A and B are two sets

$$A \approx A \times \{1\} = A'$$

$$B \approx B \times \{1\} = B'$$

$$\Rightarrow A' \cap B' = \emptyset$$

(S3)

If $|A| = \alpha$ and $|B| = \beta$

Then $|A'| = \alpha$ and $|B'| = \beta$

but A' and B' are disjoint sets.

Examples.

(i) $N_0 + N_0 = ?$ 2016,

Take the sets $A = \{2, 4, 6, \dots\}$ and $B = \{1, 3, 5, \dots\}$.

$|A| = N_0$, $|B| = N_0$ and $A \cap B = \emptyset$.

$\alpha + \beta = |(A \cup B)| = |\{1, 2, 3, 4, \dots\}| = N_0$

$\Rightarrow N_0 + N_0 = N_0$.

(ii) $N_0 + n = ?$

Take $A = \{n+1, n+2, n+3, \dots\} \approx N$

and $B = \{1, 2, 3, \dots, n\}$

$|A| = N_0$, $|B| = n$ and $A \cap B = \emptyset$.

$N_0 + n = |(A \cup B)| = |\{1, 2, 3, \dots, n, n+1, n+2, \dots\}| = N_0$

$\Rightarrow N_0 + n = N_0$.

(iii) $C + C = ?$

Take $A = [0, \frac{1}{2}]$ and $B = (\frac{1}{2}, 1]$

$|A| = c$, $|B| = c$ and $A \cap B = \emptyset$.

$C + C = |(A \cup B)| = |[0, \frac{1}{2}] \cup (\frac{1}{2}, 1]| = |[0, 1]| = c$

$\Rightarrow C + C = C$.

Cardinal Multiplication.

Let α and β be cardinal numbers and

let A and B be sets with $|A| = \alpha$ and $|B| = \beta$.

Then the product of α and β is defined as

$\alpha\beta = |A \times B|$ (Cartesian product)

V.V. Imp Examples.

① $\aleph_0 \aleph_0 = ?$

Take $A = \{1, 2, 3, \dots\}$ and $B = \{1, 2, 3, \dots\}$

$$|A| = \aleph_0, \quad |B| = \aleph_0$$

$$\aleph_0 \aleph_0 = |A \times B|$$

Now $A \times B \approx P$ as $A \approx P$ and $B \approx P$
where $P = \{1, 2, 3, \dots\}$

$$\Rightarrow \aleph_0 \aleph_0 = \aleph_0$$

(ii) $\mathbb{C} \mathbb{C} = \mathbb{C}$

Take $A = \mathbb{R}$ and $B = \mathbb{R}$.

$$|A| = |\mathbb{R}| = \mathbb{C} \quad \text{and} \quad |B| = |\mathbb{R}| = \mathbb{C}$$

$$\mathbb{C} \mathbb{C} = |A \times B| = |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}^2|$$

$$\text{Since } \mathbb{R}^2 \approx \mathbb{R} \quad \text{so} \quad |\mathbb{R}^2| = \mathbb{C}$$

$$\Rightarrow \mathbb{C} \mathbb{C} = \mathbb{C}$$

Remarks.

(i) In cardinal multiplication, the sets A and B need not be disjoint.

(ii) Cancellation laws of real numbers in general do not hold in cardinal numbers.

Counter example is $\aleph_0 \aleph_0 = \aleph_0$.

Prove the following Properties.

① $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$. 2013,

Let A, B, C be pairwise disjoint sets such that

$$|A| = \alpha, \quad |B| = \beta, \quad |C| = \gamma.$$

(55)

$$\begin{aligned}
 (\alpha + \beta) + \gamma &= |A \cup B| + |C| = |(A \cup B) \cup C| \\
 &= |A \cup (B \cup C)| \quad \because A \cup (B \cup C) = (A \cup B) \cup C \\
 &= |A| + |B \cup C| \\
 &= \alpha + (\beta + \gamma)
 \end{aligned}$$

$$\textcircled{2} \quad \alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$$

Let A, B, C be pairwise disjoint sets such that
 $|A| = \alpha, |B| = \beta, |C| = \gamma$.

$$|A \cup B| = \alpha + \beta, \quad |B \cup C| = \beta + \gamma$$

$$|A \times B| = \alpha\beta, \quad |A \times C| = \alpha\gamma$$

$$\text{If } B \cap C = \emptyset$$

$$\text{then } (A \times B) \cap (A \times C) = \emptyset$$

$$\begin{aligned}
 \text{Now } \alpha\beta + \alpha\gamma &= |(A \times B) \cup (A \times C)| \quad |A \times B| + |A \times C| \\
 &= |A \times (B \cup C)| \\
 &= \alpha(\beta + \gamma).
 \end{aligned}$$

$$\textcircled{3} \quad (\alpha\beta)\gamma = \alpha(\beta\gamma).$$

Let A, B, C be pairwise disjoint sets such that
 $|A| = \alpha, |B| = \beta, |C| = \gamma$.

$$\begin{aligned}
 \text{Now } (\alpha\beta)\gamma &= |(A \times B) \times C| \\
 &= |A \times (B \times C)| \quad \because A \times (B \times C) \simeq (A \times B) \times C \\
 &= \alpha(\beta\gamma).
 \end{aligned}$$

$$\text{If } \alpha \leq \beta, \text{ then } \alpha + \gamma \leq \beta + \gamma.$$

Let A, B, C be pairwise disjoint sets such that
 $|A| = \alpha, |B| = \beta, |C| = \gamma$.

Suppose $\alpha \leq \beta$. Then there exists a one-to-one mapping $f: A \rightarrow B$. Let $g: A \cup C \rightarrow B \cup C$ be a

(56)

mapping defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \in A \\ x & \text{if } x \in C \end{cases}$$

Clearly g is one-one as f is 1-1 and $g(x) = x$ identity function is also 1-1.

Hence $|A \cup C| \leq |B \cup C|$, $A \cap C = \emptyset$, $B \cap C = \emptyset$
 $\Rightarrow \alpha + \gamma \leq \beta + \gamma$.

(5) If $\alpha \leq \beta$, then $\alpha\gamma \leq \beta\gamma$.

Let A, B, C be pairwise disjoint sets such that
 $|A| = \alpha$, $|B| = \beta$, $|C| = \gamma$.

$$A \cap B = \emptyset, B \cap C = \emptyset, C \cap A = \emptyset.$$

$$\text{Also } |A \times B| = \alpha\beta = |B \times A|$$

$$|B \times C| = \beta\gamma = |C \times B|$$

$$|A \times C| = \alpha\gamma = |C \times A|.$$

Suppose $\alpha \leq \beta$. Then there exists one-to-one mapping $f: A \rightarrow B$. Let $g: A \times C \rightarrow B \times C$ be a mapping defined by

$$g(a, c) = (f(a), c).$$

Clearly g is one-one.

$$\text{Thus } |A \times C| \leq |B \times C|$$

$$\Rightarrow \alpha\gamma \leq \beta\gamma.$$

Exponent of Cardinal numbers.

Let α, β be two cardinal numbers and let A and B be sets with $\alpha = |A|$ and $\beta = |B|$.

(57)

then α to the power β is denoted by α^β and is defined as

$$\alpha^\beta = |A^\beta|$$

Remark.

$$C(X) = \{f: X \rightarrow \{0,1\}\} ; C(X) = P(X) \text{ --- ①}$$

$$|C(X)| = 2^{|X|}$$

$$\Rightarrow |P(X)| = 2^{|X|}$$

Example.

① Prove that $\aleph_0 c = c$.

Consider the set of integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ and the half open interval $A = [0, 1)$ such that $|\mathbb{Z}| = \aleph_0$ and $|[0, 1)| = c$.

Define a function $f: \mathbb{Z} \times A \rightarrow \mathbb{R}$ as for $n \in \mathbb{Z}$ and $x \in A$.

$$f(n, x) = n + x$$

f is one-one.

$$\text{Suppose } f(n_1, x_1) = f(n_2, x_2)$$

$$\Rightarrow n_1 + x_1 = n_2 + x_2 \longrightarrow \text{①}$$

Since $0 \leq x_1, x_2 < 1$, therefore, ① holds only when $n_1 = n_2$ and $x_1 = x_2$.

$$\Rightarrow (n_1, x_1) = (n_2, x_2)$$

$\Rightarrow f$ is one-one.

$$\Rightarrow \mathbb{Z} \times A \sim f(\mathbb{Z} \times A) = \bigcup_{n \in \mathbb{Z}} [n, n+1) = \mathbb{R}$$

$$\Rightarrow |\mathbb{Z} \times A| = c = |\mathbb{R}|$$

$$\Rightarrow \aleph_0 c = c.$$

(58)

② Let α be any infinite cardinal number.

$$\text{Then } \aleph_0 + \alpha = \alpha.$$

Suppose α is infinite cardinal number. Let A be the set such that $\alpha = |A|$.

If A is denumerable, then, $\alpha = |A| = \aleph_0$.

$$\aleph_0 + \alpha = \aleph_0 + \aleph_0 = \aleph_0 = \alpha$$

$$\Rightarrow \aleph_0 + \alpha = \alpha.$$

Now suppose A is uncountable then there exists a denumerable subset, say, B of A .

$$\text{Also } A \setminus B \approx A$$

$$\text{Now } A = B \cup (A \setminus B).$$

i.e. A can be written as disjoint union of B and $A \setminus B$ such that B is denumerable and $A \setminus B$ is non denumerable.

$$\Rightarrow |A| = |B| + |A \setminus B| \longrightarrow \textcircled{1}$$

Now A and $A \setminus B$ are non denumerable,

$$|A| = |A \setminus B| = c$$

$$\Rightarrow \alpha = c \longrightarrow \textcircled{2}$$

Also B is denumerable, $|B| = \aleph_0$.

$$\text{Thus } \textcircled{1} \Rightarrow c = \aleph_0 + c$$

$$\Rightarrow \alpha = \aleph_0 + \alpha \quad \because \text{by } \textcircled{2}$$

Example. Show that for cardinal numbers

$$\textcircled{1} \quad \alpha^{\beta+\gamma} = \alpha^{\beta} \cdot \alpha^{\gamma}.$$

$$\alpha^{\beta} = |A^B| = \text{number of functions from } B \text{ to } A$$

$$\alpha^{\gamma} = |A^C| = \text{number of functions from } C \text{ to } A$$

(59)

$\alpha^{\beta+\gamma} = |A^{B \cup C}| = \text{number of functions from } B \cup C \text{ to } A, \text{ where } B \cap C = \emptyset.$

Now for any $f: B \cup C \rightarrow A$, we can define restriction

$f_B: B \rightarrow A$ and $f_C: C \rightarrow A$ such that

$$f_B \in A^B \quad \text{and} \quad f_C \in A^C.$$

Now define $\phi: A^{B \cup C} \rightarrow A^B \times A^C$ such that

$$\phi(f) = (f_B, f_C)$$

ϕ is 1-1.

$$\text{suppose } \phi(g) = \phi(h)$$

$$\Rightarrow (g_B, g_C) = (h_B, h_C)$$

$$\Rightarrow g_B = h_B, g_C = h_C$$

$$\Rightarrow g = h$$

$$\Rightarrow \phi \text{ is 1-1.}$$

$$\text{Let } (f_1, f_2) \in A^B \times A^C.$$

$$\Rightarrow f_1: B \rightarrow A \text{ and } f_2: C \rightarrow A.$$

Now as $B \cap C = \emptyset$, we can define a function

$$g = \begin{cases} f_1(x), & \text{if } x \in B \\ f_2(x), & \text{if } x \in C \end{cases}$$

$\Rightarrow g$ is a function from $B \cup C$ to A .

$\Rightarrow g \in A^{B \cup C}$, such that

$$\phi(g) = (g_B, g_C) = (f_1, f_2)$$

$$\Rightarrow A^{B \cup C} \approx A^B \times A^C$$

$$\Rightarrow \alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$$

as required result

$$② (\alpha^B)^\gamma = \alpha^{\beta\gamma}$$

Let A, B, C be the sets such that $|A| = \alpha$, $|B| = \beta$ and $|C| = \gamma$

$$\alpha^{\beta\gamma} = |A^{B \times C}|$$

$$\text{and } (\alpha^B)^\gamma = |(A^B)^C|$$

where $A^{B \times C}$ is the collection of all functions from $B \times C$ to A , whereas $(A^B)^C$ is the collection of all functions from C to A^B .

Define $\varphi: A^{B \times C} \rightarrow (A^B)^C$, for $f \in A^{B \times C}$.

$\varphi(f) = g$ such that

$$f(b, c) = g_c(b)$$

where $g_c \in A^B$ and $g_c = g(c)$.

φ is well defined and 1-1.

Suppose $\varphi(f) = \varphi(f')$

$$\Leftrightarrow g = g'$$

$$\Leftrightarrow g(c) = g'(c), \quad \forall c \in C$$

$$\Leftrightarrow g_c = g'_c$$

$$\Leftrightarrow g_c(b) = g'_c(b)$$

$$\Leftrightarrow f(b, c) = f'(b, c), \quad \forall b \in B, c \in C$$

$$\Leftrightarrow f = f'$$

Thus φ is well defined and one-one.

φ is onto.

For this let $g \in (A^B)^C$ then there exists a mapping $g: C \rightarrow A^B$ such that

$$g(c) \in A^B.$$

Hint 1

$$f: B \times C \rightarrow A$$

$$f(b, c) \in A$$

$$g: C \rightarrow A^B$$

$$g(c) \in A^B$$

$$\text{or } g_c \in A^B$$

$$\Rightarrow g_c(b) \in A$$

Hint 2

$$f(b, c) =$$

$$g_c = g(c)$$

$$f(b, c) =$$

$$g'_c = g'(c)$$

$$\Rightarrow g'_c =$$

Let $g_c = g(c) \in A^B$

Then $g_c(b) \in A, \forall b \in B$.

Now we can define a function $f: B \times C \rightarrow A$
 $f(b, c) = g_c(b)$ such that $f \in A^{B \times C}$.

and $\phi(f) = g$.

$\Rightarrow \phi$ is onto.

$\Rightarrow \phi$ is bijective.

$$\Rightarrow |A^{B \times C}| = |(A^B)^C|$$

$$\Rightarrow \alpha^{\beta\gamma} = (\alpha^\beta)^\gamma$$

$$\textcircled{3} (\alpha\beta)^\gamma = \alpha^\gamma \beta^\gamma$$

Let A, B, C be the sets such that $|A| = \alpha$,
 $|B| = \beta$ and $|C| = \gamma$.

$$(\alpha\beta)^\gamma = |(A \times B)^C|$$

$$\text{and } \alpha^\gamma \beta^\gamma = |A^C| \cdot |B^C| = |A^C \times B^C|$$

Where $(A \times B)^C$ is the collection of all functions from C to $A \times B$, A^C the collection of all functions from C to A and B^C the collection of all functions from C to B .

Define a mapping $\phi: (A \times B)^C \rightarrow A^C \times B^C$

for $f \in (A \times B)^C$, such that

$$\phi(f) = (g, h)$$

ϕ is one-one.

$$\text{suppose } \phi(f_1) = \phi(f_2)$$

$$\Rightarrow (g_1, h_1) = (g_2, h_2)$$

$$\Rightarrow g_1 = g_2 \text{ and } h_1 = h_2$$

$$\Rightarrow f_1 = f_2$$

Hint

$$f \in (A \times B)^C$$

$$f(c) \in A \times B$$

$$g \in A^C$$

$$g(c) \in A$$

$$h \in B^C$$

$$h(c) \in B$$

(62)

$\Rightarrow \phi$ is 1-1.

ϕ is onto.

For each $(g, h) \in A^c \times B^c$, there exists a function $f \in (A \times B)^c$ such that

$$\phi(f) = (g, h)$$

$\Rightarrow \phi$ is bijective.

$$\Rightarrow |(A \times B)^c| = |A^c \times B^c|$$

$$\Rightarrow (\alpha\beta)^\gamma = \alpha^\gamma \beta^\gamma$$

④ If $\alpha \leq \beta$, then $\alpha^\gamma \leq \beta^\gamma$.

Ordered Sets and Lattices

Partial Order.

Suppose R is a relation on a set S satisfying the following three properties.

1. Reflexive i.e. ~~$a \neq b$~~ for $a \in S$, aRa .

2. Antisymmetric i.e. if aRb and bRa then $a=b$.

3. Transitive i.e. if aRb and bRc then aRc .

When R is called a partial order or R is said

to define a partial ordering of S . The set S

with the partial ordering R is called partial

ordered set.

The partial ordering relation is denoted by $\preceq$,

and the ordered set is denoted by $(S, \preceq)$.

" $a \preceq b$ " is read as "a preceeds b".

" $b \succeq a$ " is read as "b succeeds a".

Examples.

1. The relation $\subseteq$ of set inclusion is partial ordering of $\mathcal{P}$ where $\mathcal{P}$ is any collection of sets.

2. The relation $|$ of divisibility on +ive integers is partial ordering.

3. The relation $\leq$ on the set of integers is partial ordering.

Dual Order.

Let $\preceq$ be any partial ordering of a set S . The

relation $\succeq$, that is "a succeeds b" is also a partial

ordering of S ; it is called dual order.

Ordered Subset:

Let A be a subset of an ordered set S , and suppose $a, b \in A$. Then the order in S induces an order in A in the following way:
 $a \leq b$ as elements of A whenever $a \leq b$ as elements of S .

If R is a partial ordering of S , then the relation $R_A = R \cap (A \times A)$ is called partial ordering of A called the induced order on A . (The subset A with the induced order is called an ordered subset of S .)

Example:

Let $R = \{ (1,1), (1,2), (1,3), (1,4), \dots, (1,10), (2,2), (2,4), (2,6), (2,8), (2,10), (3,3), (3,6), (3,9), (4,4), (4,8), (5,5), (5,10), (6,6), (7,7), (8,8), (9,9), (10,10) \}$

be a partial ordering and $A = \{ 2, 4, 6, 8, 10 \}$. Then $R_A = \{ (2,2), (2,4), (2,6), (2,8), (2,10), (4,4), (4,8), (6,6), (8,8), (10,10) \}$.

Quasi - order.

Suppose $\leq$ is a relation on a set S satisfying the following two properties:

(i) Irreflexive i.e. for any $a \in S$, $a \not\leq a$.

(ii) Transitive i.e. if $a \leq b$, and $b \leq c$, then $a \leq c$.

Then $\leq$ is called a quasi order on S .

There is a close relationship between partial orders and quasi-orders.

comparability.

Suppose a and b are distinct elements in a partially ordered set S . We say a and b are comparable if $a \leq b$ or $b \leq a$.

one of them precedes the other.

Linearly Ordered Set.

Let $(S, \leq)$ be a partial ordered set. Then S is said to be a linearly ordered set if every pair of distinct elements of S are comparable. i.e. for $a, b \in S$ either $a \leq b$ or $b \leq a$.

Remark.

A partial ordered set may or may not be a linearly ordered set but a subset A of S can be linearly ordered subset.

Chain.

If a subset A of a partial ordered set S is linearly ordered, then it is called chain.

Example.

Let $P = \{1, 2, 3, \dots\}$ be ordered by divisibility.

State whether each of the following is a chain in P .

(66)

(a) $A = \{24, 2, 6\}$

Since 2 divides 6 which divides 24, A is a chain in P.

(b) $B = \{3, 15, 5\}$

Since 3 and 5 are non comparable, B is not a chain in P.

(c) $C = \{2, 8, 32, 4\}$

Since $2 \leq 4 \leq 8 \leq 32$, that is, $2|4|8|32$ where $|$ means divides, C is a chain in P.

(d) $D = \{7\}$

Any set consisting of one element is linearly ordered, D is a chain in P.

(e) $E = \{15, 5, 30\}$

Since $5 \leq 15 \leq 30$, E is a chain in P.

(f) $P = \{1, 2, 3, \dots\}$

P is not linearly ordered as 2 and 3 are not comparable so P itself is not a chain in P.

Product order.

Suppose S and T are ordered sets. Then $(a, b) \leq (c, d)$ iff $a \leq c$ and $b \leq d$, is an order relation on the product set $S \times T$, called the product order.

Now we show that this ordering is partial.

Reflexive.

Since $a = a$ and $b = b$, $a \leq a$ and $b \leq b$.

Hence $(a, b) \leq (a, b)$ and $\leq$ is reflexive.

ii) Antisymmetric.

Suppose $(a, b) \leq (c, d)$ and $(c, d) \leq (a, b)$. Then

$a \leq c$ and $b \leq d$ and $c \leq a$ and $d \leq b$.

Thus $a = c$ and $b = d$. Hence $(a, b) = (c, d)$

and $\leq$ is antisymmetric.

iii) Transitive.

Suppose $(a, b) \leq (c, d)$ and $(c, d) \leq (e, f)$. Then

$a \leq c$ and $b \leq d$ and $c \leq e$ and $d \leq f$.

By transitivity of $\leq$, we have $a \leq e$ and

$b \leq f$.

Thus $(a, b) \leq (e, f)$, and $\leq$ is transitive.

Hence the product relation is partial order.

Lexicographical order.

Suppose S and T are ordered sets. Then,

$(a, b) \leq (c, d)$ if $a < c$ or if $a = c$ and $b \leq d$, is an

order relation on the product set $S \times T$, called

the lexicographical order or dictionary order.

Now we show that this ordering is partial.

(a) $A = \{24, 2, 6\}$

Since 2 divides 6 which divides 24, A is a chain in P .

(b) $B = \{3, 15, 5\}$

Since 3 and 5 are non comparable, B is not a chain in P .

(c) $C = \{2, 8, 32, 4\}$

Since $2 \leq 4 \leq 8 \leq 32$, that is, $2|4|8|32$ where $|$ means divides, C is a chain in P .

(d) $D = \{7\}$

Any set consisting of one element is linearly ordered, D is a chain in P .

(e) $E = \{15, 5, 30\}$

Since $5 \leq 15 \leq 30$, E is a chain in P .

(f) $P = \{1, 2, 3, \dots\}$

P is not linearly ordered as 2 and 3 are not comparable so P itself is not a chain in P .

Product order.

Suppose S and T are ordered sets. The

$(a, b) \leq (c, d)$ iff $a \leq c$ and $b \leq d$, is an order relation on the product set $S \times T$ called the product order.

Now we show that this ordering is partial.

(67)

reflexive.

Since $a = a$ and $b = b$, $a \leq a$ and $b \leq b$.

Hence $(a, b) \leq (a, b)$ and $\leq$ is reflexive.

Antisymmetric.

Suppose $(a, b) \leq (c, d)$ and $(c, d) \leq (a, b)$. Then

$a \leq c$ and $b \leq d$ and $c \leq a$ and $d \leq b$.

Thus $a = c$ and $b = d$. Hence $(a, b) = (c, d)$

and $\leq$ is antisymmetric.

iii) Transitive.

Suppose $(a, b) \leq (c, d)$ and $(c, d) \leq (e, f)$. Then

$a \leq c$ and $b \leq d$ and $c \leq e$ and $d \leq f$.

By transitivity of $\leq$, we have $a \leq e$ and $b \leq f$.

Thus $(a, b) \leq (e, f)$, and $\leq$ is transitive.

Hence the product relation is partial order.

Lexicographical order.

Suppose S and T are ordered sets. Then,

$(a, b) < (c, d)$ $\left\{ \begin{array}{l} \text{if } a < c \\ \text{or if } a = c \end{array} \right.$ and $b < d$, is an

order relation on the product set $S \times T$, called

the lexicographical order or dictionary order.

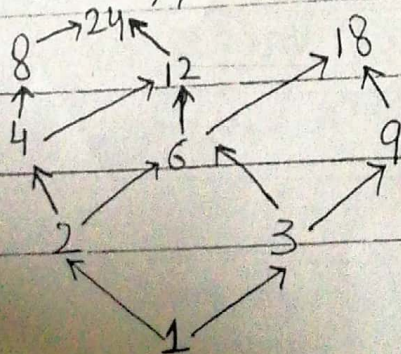
Now we show that this ordering is partial.

Immediate Predecessor and Immediate Successor.
 Let S be a poset and suppose $a, b \in S$, a is immediate predecessor of b or b is immediate successor of a if $a < b$ but no element in S lies between a and b i.e. there does not exist any $c \in S$ such that $a < c < b$.
 It is denoted as $a \ll b$.

Hasse Diagram.

The Hasse diagram of a finite poset S is a graphical representation of S as follows. The elements of S are represented by points in the plane and there is directed line drawn from a to b , whenever $a \ll b$ in S .
 For example $A = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$.

$a < b$ iff $a \mid b$.



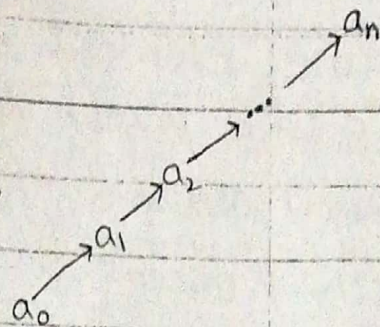
(69)

Remark.

If S is linearly ordered set
then Hasse diagram is simply
consists of one path.

$$S = \{a_0, a_1, a_2, \dots, a_n\}$$

$$a_0 < a_1 < a_2 < \dots < a_n.$$



Example.

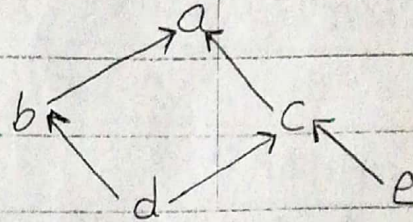
Write order for the following diagram.

$$d < b, d < c, e < c,$$

$$b < a, d < a, c < a, e < a.$$

$$\{(d, b), (d, c), (e, c), (b, a),$$

$$(d, a), (c, a), (e, a)\}.$$



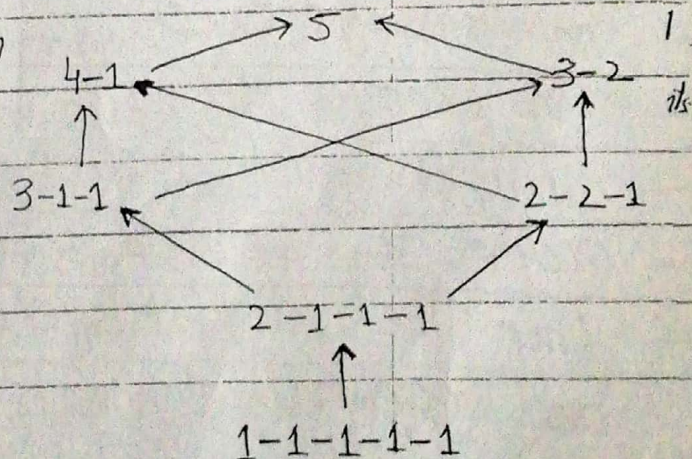
Example.

Let n be a +ive integer and S be the set
of all partitions of 5 .

Let $P_1, P_2 \in S$. Then $P_1 \leq P_2$ if the integers in P_1
can be added to obtain the integers in P_2 .

The collection containing the partitions of 5 is
 $\{5, 4-1, 3-2, 3-1-1, 2-2-1, 2-1-1-1, 1-1-1-1-1\}$.

The Hasse diagram
is given as



Minimal Element.

Let S be a poset. An element $a \in S$ is called minimal element of S if no element of S strictly precedes a .

$$\text{i.e. } x \leq a \Rightarrow x = a.$$

Maximal Element.

Let S be a poset. An element $b \in S$ is called a maximal element of S if no element of S strictly succeeds b .

$$\text{i.e. } x \geq b \Rightarrow x = b.$$

Remarks.

- ① A poset may have more than one maximal or minimal elements.

For example $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

$$x \leq y \Leftrightarrow x|y.$$

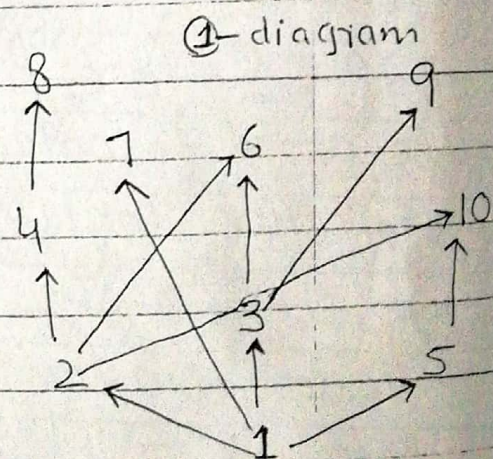
The Hasse diagram is given as

- ② The minimal and maximal elements not always exist.

For example set of integers with respect to

$$\text{order } \leq \text{ i.e. } x \leq y \Leftrightarrow x \leq y.$$

Then the set of integers has no minimal and maximal elements.



(71)

Element.

Let S be a poset. An element $a \in S$ is said to be 1st element if it precedes every element in S .

$$a \leq x \quad \forall x \in S.$$

Element.

Let S be a poset. An element $b \in S$ is said to be last element if it succeeds every element in S .

$$b \geq x \quad \forall x \in S.$$

Remarks.

S can have at most one 1st element and at most one last element.

A first element is always minimal and last is always maximal but converse may not be true.

S may have neither last nor first element.

For example,

$$P = \{1, 2, 3, \dots\} \rightarrow \text{1st element} = 1, \text{ last element} = \text{no.}$$

$$A = \{\dots, -3, -2, -1\} \rightarrow \text{1st element} = \text{no}, \text{ last element} = -1$$

The set of integers have not 1st and last elements.

Existence of 1st and last element implies existence of minimal and maximal element but

converse does not hold in general.

For example $A = \{1, 2, \dots, 10\}$ $x \leq y \Leftrightarrow x \leq y$.

(72)

Then 1 is 1st element but S does not have any last element although the maximal elements of S are $\{6, 7, 8, 9, 10\}$.

⑤ If S is linearly ordered set then it must contain minimal element which is also 1st element and it must contain maximal element which is last element.

Upper bound.

Let S be a poset and $A \subseteq S$. An element M is called an upper bound of A if M succeeds every element of A .

$$\text{i.e. } x \leq M \quad \forall x \in A.$$

Lower bound.

Let S be a poset and $A \subseteq S$. An element m is called a lower bound of A if m preceeds every element of A .

$$\text{i.e. } m \leq x \quad \forall x \in A.$$

Supremum.

If an upper bound of A preceeds every other upper bound of A , then it is called the supremum of A and is denoted by $\text{Sup}(A)$ or $\text{LUB}(A)$.

Infimum.

If a lower bound of A succeeds every other lower bound of A , then it is called

(73)

maximum of A and is denoted by $\text{Inf}(A)$ or $\text{GLB}(A)$.

examples.

Let $S = \{a, b, c, d, e, f\}$ be ordered set as shown in Fig.

and let $A = \{b, c, d\}$.

lower bounds of $A = \{a, b\}$

only a and b

precede every element of A .

upper bound of $A = \{e, f\}$,

only e and f succeed every element of A .

since b succeeds a , $\text{Inf}(A) = b$, $b \in S$.

so e and f are non-comparable so $\text{sup}(A)$

does not exist.

Let $S = \{1, 2, 3, \dots, 8\}$ be ordered set as shown in Fig.

and let $A = \{4, 5, 7\}$.

upper bound of $A = \{8\}$

since 7 does not precede

7 is not a lower

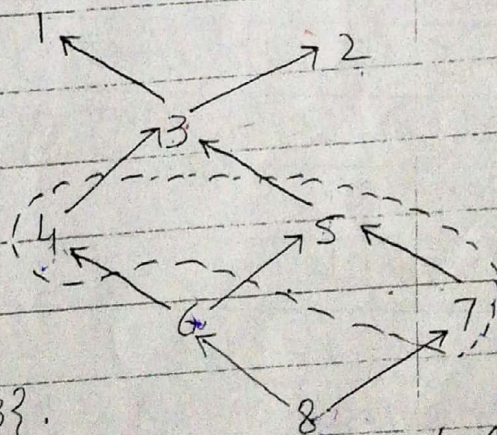
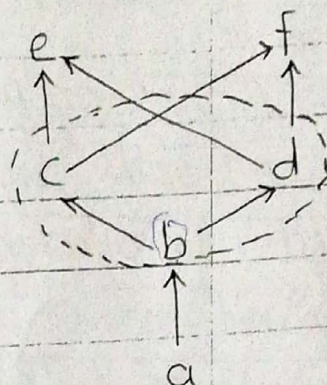
bound.

upper bound of $A = \{1, 2, 3\}$.

$\text{Inf}(A) = 8$ as 8 is the only lower bound of A .

$\text{sup}(A) = 3$ as 3 precedes the other upper

bounds, 1 and 2 , of A .



(74)

③ Consider the $\mathbb{Q}$ of rational numbers and its subset $B = \{x \in \mathbb{Q} : x > 0 \text{ and } 2 < x^2 < 3\}$. i.e. B consists of those rational numbers which lie between $\sqrt{2}$ and $\sqrt{3}$ on the real line $\mathbb{R}$.

Then B has an infinite number of upper and lower bounds but $\inf(B)$ and $\sup(B)$ do not exist. In other words, B has no least upper bound and no greatest lower bound.

Existence of $\sup(a, b)$ and $\inf(a, b)$.

Let S be an ordered set and $a, b \in S$.

If S is linearly ordered, then $\sup(a, b)$ and $\inf(a, b)$ clearly exist.

If $a \leq b$ then $\sup(a, b) = b$ and $\inf(a, b) = a$.

Remark.

If S is not linearly ordered, then $\sup(a, b)$ and $\inf(a, b)$ may or may not exist in general.

Example.

① Let $P = \{1, 2, 3, \dots\}$ be the set of natural numbers and $x \leq y \Leftrightarrow x \mid y$.

Let $a, b \in P$.

Since $\gcd(a, b)$ divides a, b and a, b divide $\text{lcm}(a, b)$.

(75)

and
then
and
al
inf(a,b) and sup(a,b) exist for any pair
of elements of P ordered by divisibility.

and
not
Let S be a non empty set with at least
two elements, and let P(S) be the power
set of S ordered by inclusion.

Let A and B be any two elements of P(S).

Then sup(A,B) and inf(A,B) do exist.

b) and i.e. sup(A,B) = A ∪ B

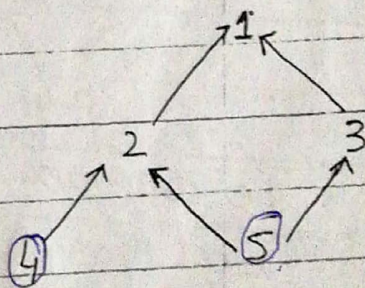
and inf(A,B) = A ∩ B.

Question.

Consider the ordered set A as shown in Fig

@ Find all minimal and
sup maximal elements of A.

st in @ Does A have a 1st
element or a last element?



Solution.

natur @ Minimal elements of A are 4, 5 as no element
strictly precedes 4 or 5.

divi Maximal element of A is 1 as no element
strictly succeeds 1.

@ A has no first element. Although 4 and 5
are minimal elements of A but neither precedes
the other.

(76)

Last element of A is 1 as ^{1 succeeds} every element of A .

Question.

Consider the ordered set A . For each $a \in A$, let $p(a)$ denote the set of predecessors of a ,

$$\text{i.e. } p(a) = \{x : x \leq a\}.$$

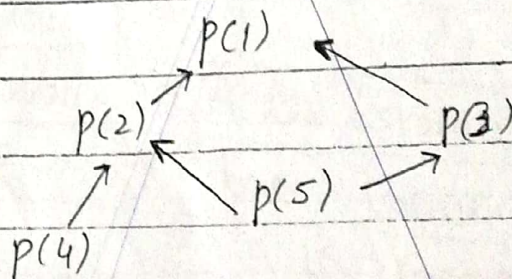
Let $P(A) = \{p(a) : a \in A\}$ denote the collection of all predecessor sets of A , and let $P(A)$ be ordered by set inclusion.

The elements of $P(A)$ are

$$p(1) = \{1, 2, 3, 4, 5\}, \quad p(2) = \{2, 4, 5\}, \quad p(3) = \{3, 5\}$$

$$p(4) = \{4\}, \quad p(5) = \{5\}.$$

The Hasse diagram of $P(A)$ is



The diagram of A and $P(A)$ are same except for labelling of the vertices.

Problem.

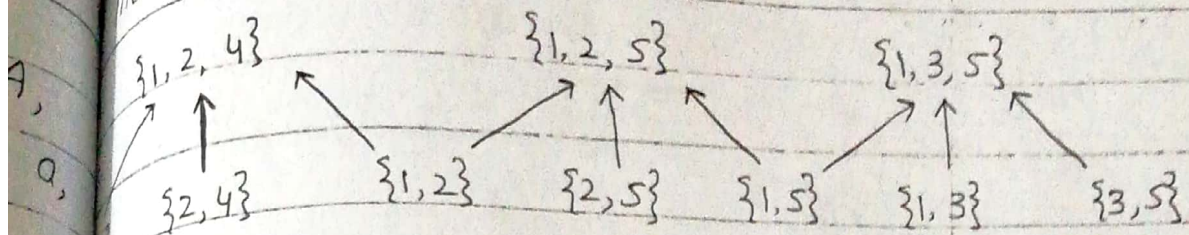
Let $L(A)$ denote the collection of all chains in A with 2 or more elements, and let be ordered by set inclusion.

Draw the Hasse diagram of $L(A)$.

The elements of $L(A)$ are

$$\{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 5\}, \{1, 2\}, \{1, 4\}, \{1, 3\}, \{1, 5\}, \{2, 4\}$$

of A. The Hasse diagram of $L(A)$ is given as



Similarity Mapping.

Suppose X and Y are posets. A one-to-one ^{bijection} mapping $f: X \rightarrow Y$ is called a similarity mapping

from X into Y if f preserves the order relation.

i.e. for any pair $a, b \in X$ the condition
 $a \leq b$ in X iff $f(a) \leq f(b)$ in Y holds.

Order Isomorphic.

Two ordered sets X and Y are said to be order-isomorphic i.e. $X \cong Y$, if there exists a

one-to-one ^{and onto} mapping $f: X \rightarrow Y$ which preserves the order relations. Such a mapping f is

called an order isomorphism between X and Y .

i.e. f is one-one and onto.

and $a \leq b$ iff $f(a) \leq f(b)$.

Example.

Let $P = \{1, 2, 3, \dots\}$ be the set of +ive integers and $E = \{2, 4, 6, \dots\}$ be the set of +ive even integers.

Define a function $f: P \rightarrow E$ as

$$f(x) = 2x$$

$$x \leq y \Leftrightarrow x \leq y$$

Clearly f is 1-1 and onto

$$\Leftrightarrow x \leq y$$

$$\Leftrightarrow 2x \leq 2y$$

$$\Leftrightarrow f(x) \leq f(y)$$

$\Rightarrow f$ preserves order.

$$\Rightarrow P \simeq E.$$

Theorem.

Suppose S is linearly ordered and $T \simeq S$.

Then T is linearly ordered.

Proof.

Since $T \simeq S$, there exists a function $f: S \rightarrow T$ which is one-one and onto and preserves order. To prove T is linearly ordered, let $x, y \in T$.

We show that x and y are comparable.

Since f is onto, there exist $x_1, y_1 \in S$ such

that $f(x_1) = x$

$$f(y_1) = y$$

Since $x_1, y_1 \in S$ and S is linearly ordered so x_1 and y_1 are comparable.

Since f is similarity mapping so $f(x_1)$ and $f(y_1)$ are comparable.

$\Rightarrow x$ and y are comparable.

Thus T is linearly ordered.

Theorem.

If S is order isomorphic to T , then S is equivalent to T ; that is, if $S \cong T$ then $|S| = |T|$.

Proof.

Let $S \cong T$ then there exists a bijective mapping $f: S \rightarrow T$ which preserves order.

$$\Rightarrow S \cong T$$

$$\Rightarrow |S| = |T|$$

Remark:

The converse of above theorem may not hold in general.

For example, consider the set $P = \{1, 2, 3, \dots\}$ and $A = \{-1, -2, -3, \dots\}$ with the usual ordering $\leq$.

Define a mapping $f: P \rightarrow A$ such that

$$f(n) = -n \quad \forall n \in P$$

Clearly f is 1-1 and onto.

$$\Rightarrow P \cong A$$

As P is equipotent to A i.e. $|P| = |A|$.

But for $n_1 \leq n_2$

$$-n_1 \geq -n_2$$

$$\Rightarrow f(n_1) \geq f(n_2)$$

$\Rightarrow f$ does not preserve order.

$$\Rightarrow P \not\cong A.$$

Theorem.

The relation of order isomorphism between ordered sets is an equivalence relation.

Proof.

Reflexive.

For any ordered set A , the identity mapping $f: A \rightarrow A$ defined by

$$f(a) = a \quad \forall a \in A.$$

is bijective and also preserves the order,

$$\text{i.e. } a_1 \leq a_2 \Leftrightarrow f(a_1) \leq f(a_2)$$

$$\Rightarrow A \simeq A.$$

$\Rightarrow \simeq$ is reflexive.

Symmetric.

Let $A \simeq B$. Then there exists a bijective map

$f: A \rightarrow B$ such that

$$a_1 \leq a_2 \Leftrightarrow f(a_1) \leq f(a_2).$$

Since f is bijective, $f^{-1}: B \rightarrow A$ exists and is also bijective.

Let $b_1 \leq b_2$. Since f is onto, there exist a_1 such that $f(a_1) = b_1$ and a_2 such that $f(a_2) = b_2$.

$$\text{When } f(a_1) = b_1 \Rightarrow a_1 = f^{-1}(b_1)$$

$$f(a_2) = b_2 \Rightarrow a_2 = f^{-1}(b_2)$$

Since f is similarity mapping so

$$a_1 \leq a_2 \Leftrightarrow f^{-1}(b_1) \leq f^{-1}(b_2).$$

$\Rightarrow B \simeq A. \Rightarrow \simeq$ is symmetric.

(81)

transitive.

Let $A \simeq B$ and $B \simeq C$. Then there exist bijective mappings $f: A \rightarrow B$ and $g: B \rightarrow C$ such that

$$a_1 \leq a_2 \Leftrightarrow f(a_1) \leq f(a_2) \text{ and } b_1 \leq b_2 \Leftrightarrow g(b_1) \leq g(b_2).$$

Then the composition function $g \circ f: A \rightarrow C$ is bijective.

Let $a_1 \leq a_2$. Since f is similarity mapping

so $f(a_1) \leq f(a_2)$. Also f is onto, for $b_1, b_2 \in B$ there exist $a_1, a_2 \in A$ such that $f(a_1) = b_1, f(a_2) = b_2$.

Also g is similarity mapping so

$$\begin{aligned} f(a_1) \leq f(a_2) &\Leftrightarrow g(f(a_1)) \leq g(f(a_2)) \\ &\Leftrightarrow g \circ f(a_1) \leq g \circ f(a_2). \end{aligned}$$

$$\Rightarrow a_1 \leq a_2 \Leftrightarrow g \circ f(a_1) \leq g \circ f(a_2).$$

$$\Rightarrow A \simeq C.$$

$$\Rightarrow \simeq \text{ is transitive.}$$

Hence $\simeq$ is an equivalence relation.

Theorem.

Suppose $f: S \rightarrow T$ is an ordered isomorphism between ordered sets S and T . Then $a \in S$ is a first, last, minimal, or maximal iff. $f(a)$ is, respectively, a first, last, minimal, or maximal element of T .

Proof.

Let $a \in S$ be the 1st element of S .

$$\text{i.e. } a \leq x, \quad \forall x \in S \longrightarrow \textcircled{1}.$$

Let $y \in T$. Since $f: S \rightarrow T$ is onto so there

(82)

exists some $x \in S$ such that

$$f(x) = y \longrightarrow (2)$$

As f is ordered isomorphism so from (1)

$$f(a) \leq f(x) \quad \forall x \in S.$$

$$\Rightarrow f(a) \leq y \quad \forall y \in T \quad \because \text{using (2)}$$

Which shows that $f(a)$ is 1st element of T .

Conversely suppose that $f(a)$ is 1st element of T .

$$\text{i.e. } f(a) \leq f(x) \quad \forall f(x) \in T.$$

Since f is similarity mapping, therefore,

$$a \leq x \text{ as } f: T \rightarrow S \text{ is also}$$

similarity mapping.

This shows that $a \in S$ is 1st element of S .

Let $a \in S$ be the last element of S .

$$\text{i.e. } x \leq a, \quad \forall x \in S \longrightarrow (3)$$

Let $y \in T$. Since $f: S \rightarrow T$ is onto so there

exists some $x \in S$ such that

$$f(x) = y \longrightarrow (4)$$

As $f: S \rightarrow T$ is ordered isomorphism so from (3),

$$f(x) \leq f(a), \quad \forall f(x) \in T.$$

$$\Rightarrow y \leq f(a), \quad \forall y \in T \quad \because \text{using (4)}$$

Which shows that $f(a)$ is last element of

T .

(83)

Conversely suppose that $f(a)$ is last element of T .

$$\text{i.e. } f(a) \geq f(x), \forall f(x) \in T.$$

Since $f: S \rightarrow T$ is ordered isomorphism, therefore,

$f^{-1}: T \rightarrow S$ is ordered isomorphism.

$$a \geq x, \forall x \in S$$

which shows that $a \in S$ is last element of S .

Let $a \in S$ be the minimal element of S .

$$\text{i.e. } x \leq a \Rightarrow x = a \longrightarrow \textcircled{5}$$

On contrary suppose that $f(a)$ is not minimal element of T . Then for any $y \in T$ there exists some $x \in S$ such that

$$f(x) = y \longrightarrow \textcircled{6}$$

Since $f(a)$ is not minimal element of T .

Therefore, $y < f(a)$.

$$\Rightarrow f(x) < f(a) \quad \because \text{using } \textcircled{6}.$$

Since $f: S \rightarrow T$ is ordered isomorphism.

Therefore, $x < a, \forall x \in S$.

which is contradict to our supposition $\because$ by $\textcircled{5}$

Hence $f(a)$ is minimal element of T .

Conversely suppose that $f(a)$ is minimal element of T .

$$\text{i.e. } f(x) \leq f(a) \Rightarrow f(x) = f(a); \forall f(x) \in T \longrightarrow \textcircled{7}$$

(84)

On contrary suppose that 'a' is not minimal element of S . Then there exists some $x \in S$ such that $x < a$; $\forall x \in S$.

Since $f: S \rightarrow T$ is ordered isomorphism.

Therefore, $f(x) < f(a)$, $f(x) \in T$.

which is contradiction to our supposition.

(by ⑦).

Hence ~~for~~ a is minimal element of S .

Let $a \in S$ be the maximal element of S .

i.e. $a \leq x \Rightarrow a = x \longrightarrow$ ⑧ $x \in S$

Suppose on contrary that $f(a)$ is not maximal element of T . For any $y \in T$ there exists some $x \in S$ such that

$$f(x) = y$$

Since $f(a)$ is not maximal element of T

Therefore, $f(a) < y$

$$\Rightarrow f(a) < f(x).$$

$f(x) \in T$.

Since $f: S \rightarrow T$ is ordered isomorphism

Therefore, $a < x$, $x \in S$.

which is contradiction to our supposition.

Hence $f(a)$ is maximal element of T .

Conversely suppose that $f(a)$ is maximal element of T .

(85)

i.e. $f(a) \leq f(x) \Rightarrow f(a) = f(x)$, $f(x) \in T$.

Suppose to the contrary that a is ^⑨ not maximal element of S . Then there exists some $x \in S$ such that $a < x$; $x \in S$.

Since $f: S \rightarrow T$ is ordered isomorphism.

Therefore, $f(a) < f(x)$; $f(x) \in T$.

which is contradiction to our supposition. $\therefore$ by ⑨

Hence ' a ' is maximal element of S .

Lattice.

Let L be a non empty set closed under two binary operations called meet ($\wedge$) and join ($\vee$).

Then L is called lattice if the following axioms hold.

i) Commutative Law.

$$a \wedge b = b \wedge a \quad ; \quad a \vee b = b \vee a$$

ii) Associative Law.

$$(a \wedge b) \wedge c = a \wedge (b \wedge c) \quad ; \quad (a \vee b) \vee c = a \vee (b \vee c)$$

iii) Absorption Law.

$$a \wedge (a \vee b) = a \quad ; \quad a \vee (a \wedge b) = a$$

Where a, b, c are elements of L .

Lattice is denoted by $(L, \wedge, \vee)$.

Example.

Let C be collection of sets which is closed under union and intersection.

Then $(C, \cap, \cup)$ is a lattice.

Solution.

① ~~Associative~~ Commutative law.

For any sets A, B .

$$A \cup B = B \cup A ; A \cap B = B \cap A.$$

② Associative law.

For any sets A, B and C .

$$A \cap (B \cap C) = (A \cap B) \cap C.$$

$$A \cup (B \cup C) = (A \cup B) \cup C.$$

③ Absorption law.

For any sets A, B .

$$A \cup (A \cap B) = A ; A \cap (A \cup B) = A.$$

Duality.

The dual of any statement in a lattice $(L, \wedge, \vee)$ is defined to be the statement that is obtained by interchanging $\wedge$ and $\vee$.

For example the dual of $a \wedge (b \vee a) = a$ is $a \vee (b \wedge a) = a$.

Note.

The dual of each axiom of a lattice is also an axiom.

Principle of duality.

The dual of any theorem in a lattice is also a theorem.

Idempotent law.

0111

$$a \wedge a = a$$

$$(ii) a \vee a = a.$$

(87)

Proof.

$$(i) \quad a \wedge a = a \wedge [a \vee (a \wedge b)] \quad \text{by absorption law}$$

$$\text{Let } c = a \wedge b.$$

$$\text{Then } a \wedge a = a \wedge (a \vee c).$$

$$= a \quad \text{by Absorption law}$$

$$(ii) \quad a \vee a = a \vee [a \wedge (a \vee b)] \quad \text{by Absorption law}$$

$$\text{Let } c = a \vee b.$$

$$\text{Then } a \vee a = a \vee (a \wedge c).$$

$$= a. \quad \text{by Absorption law}$$

Order in Lattice.

$$\text{Let } a, b \in (L, \wedge, \vee).$$

$$\text{Then } a \leq b \iff a \wedge b = a$$

$$\text{or } a \leq b \iff a \vee b = b$$

Theorem.

Let L be a lattice. Then

$$(i) \quad a \wedge b = a \quad \text{iff} \quad a \vee b = b.$$

$$(ii) \quad \text{The relation } a \leq b \text{ defined by } a \wedge b = a \\ \text{or } a \vee b = b \text{ is a partial order on } L.$$

Proof.

$$(i) \quad \text{Suppose } a \wedge b = a. \quad \longrightarrow \quad \textcircled{1}$$

By Absorption law, we have

$$b = b \vee (b \wedge a)$$

$$= b \vee (a \wedge b) \quad \text{by commutative law}$$

$$= b \vee a \quad \text{by } \textcircled{1}$$

$$= a \vee b \quad \text{by commutative law}$$

(88)

conversely suppose that $a \vee b = b \rightarrow (2)$
 By absorption law, we have,

$$a = a \wedge (a \vee b)$$

$$= a \wedge b \quad \text{by (2)}$$

Thus $a \wedge b = a$ iff $a \vee b = b$.

(ii) To show the relation $\leq$ is partial order,
 we show that, $\leq$ is reflexive, antisymmetric
 and transitive.

Reflexive.

For any a in L , we have

$$a \wedge a = a \quad \text{and} \quad a \vee a = a \quad \because \text{by idempotent law.}$$

$$\Rightarrow a \leq a.$$

$$\Rightarrow \leq \text{ is reflexive.}$$

Antisymmetric.

Suppose $a \leq b$ and $b \leq a$.

Then $a \wedge b = a$ and $b \wedge a = b \rightarrow (3) \because \text{by order in lattice}$

$$\text{Therefore, } a = a \wedge b$$

$$= b \wedge a$$

$$= b$$

$\because \text{by (3)}$

$$\Rightarrow a = b$$

$$\Rightarrow \leq \text{ is antisymmetric.}$$

Transitive.

Suppose $a \leq b$ and $b \leq c$.

Then $a \wedge b = a$ and $b \wedge c = b \rightarrow (4) \because \text{by in } L$

(89)

$$a = a \wedge b$$

$$= a \wedge (b \wedge c) \quad \because \text{by (4)}$$

$$= (a \wedge b) \wedge c \quad \text{by associative law.}$$

$$= a \wedge c \quad \because a \wedge b = a.$$

$$\Rightarrow a = a \wedge c$$

$$\Rightarrow a \leq c$$

$\Rightarrow \leq$ is transitive.

Hence, ' $\leq$ ' is a partial order on L .

Theorem.

Let P be a partially ordered set such that the $\inf(a, b)$ and $\sup(a, b)$ exist for any $a, b \in P$.

Then $(P, \wedge, \vee)$ is a lattice, where

$$a \wedge b = \inf(a, b)$$

$$\text{and } a \vee b = \sup(a, b).$$

Furthermore, the partial order on P induced by the lattice is the same as the original partial order on P .

Proof.

First we show that P is a lattice.

i) Commutative law.

Let $a, b \in P$. Then

$$a \wedge b = \inf(a, b)$$

$$= \inf(b, a).$$

$$= b \wedge a.$$

(90)

$$\begin{aligned} \text{and } a \vee b &= \sup(a, b) \\ &= \sup(b, a) \\ &= b \vee a. \end{aligned}$$

(ii) Associative law.

For $a, b, c \in P$,

$$\text{Let } b \wedge c = \inf(b, c) = d$$

$$\begin{aligned} \text{Then } a \wedge (b \wedge c) &= \inf(a, d) \\ &= \inf(a, b, c) \end{aligned}$$

$$\text{and } (a \wedge b) \wedge c = \inf(a, b, c).$$

$$\Rightarrow (a \wedge b) \wedge c = a \wedge (b \wedge c).$$

Similarly,

$$(a \vee b) \vee c = a \vee (b \vee c).$$

(iii) Absorption law

$$\text{Let } a \vee b = \sup(a, b) = d \Rightarrow a \leq d$$

$$\text{Then } a \wedge (a \vee b) = \inf(a, d)$$

$$\Rightarrow a \wedge (a \vee b) = a$$

$\because a \leq d$ then $\inf(a, d) = a$

Similarly,

$$a \vee (a \wedge b) = a.$$

Hence P is a lattice.

Now, let P' be the partial ordering induced by L .

$$\text{i.e. } a \leq b \Leftrightarrow a \wedge b = a$$

$$\Leftrightarrow \inf(a, b) = a$$

$$\Rightarrow a \leq b \Leftrightarrow \inf(a, b) = a.$$

(91)

Similarly,

$$a \leq b \Leftrightarrow a \vee b = b$$

$$a \leq b \Leftrightarrow \text{Sup}(a, b) = b$$

$$a \leq b \Leftrightarrow \text{Sup}(a, b) = b$$

Hence the partial order on P induced by the set lattice is the same as the original partial order on P .

Definition of partial order on lattice.

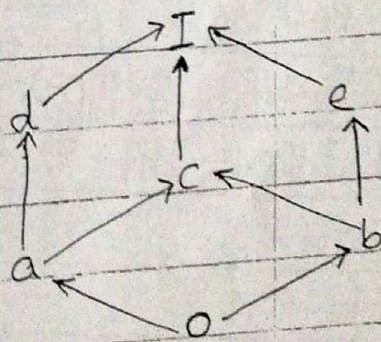
A lattice is a partial ordered set in which $a \wedge b = \inf\{a, b\}$ and $a \vee b = \sup\{a, b\}$ exist for any pair of elements a and b .

Example.

Which of the following partially ordered sets are lattices?

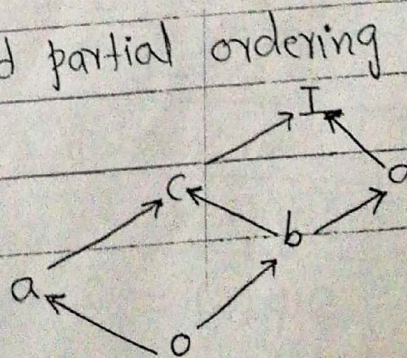
1) Let $S = \{0, a, b, c, d, e, I\}$ and partial ordering on S is given as

In this structure the supremum and infimum of each pair of a set S exist so this partially ordered set is lattice.



2) Let $S = \{0, a, b, c, d, I\}$ and partial ordering on S is given as

In this case the supremum and infimum

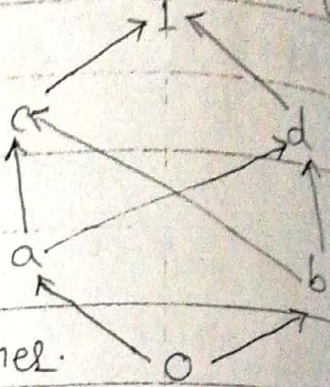


(92)

for each pair of S exist so this set is lattice.

③ Let $S = \{0, a, b, c, d, I\}$ and partial ordering on S is given by

This structure ~~has~~ is not lattice because $\{a, b\}$ has three upper bounds, c, d and I and no one of them precede the other. that is $\sup\{a, b\}$ does not exist.

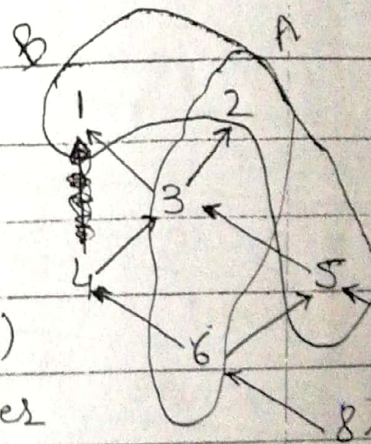


Question.

Let $S = \{1, 2, 3, \dots, 8\}$ be ordered as in Fig and let (1) $A = \{2, 3, 6\}$ (2) $B = \{1, 2, 5\}$ are subsets of S .

(a) Find the upper and lower bounds of A and B .

(b) Identify $\sup(A)$, $\sup(B)$ and $\inf(A)$, $\inf(B)$ if either exists.



Solution.

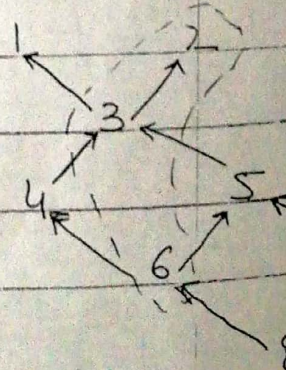
(1) upper bound of

$$A = \{2, 3\}$$

Lower bound of $A = \{8, 6\}$.

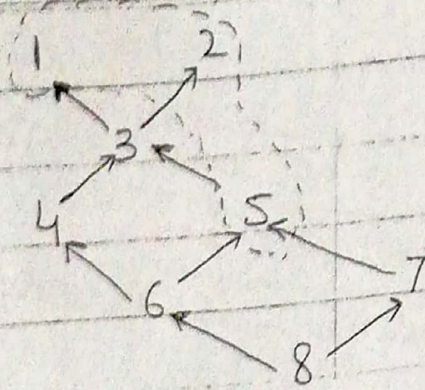
$$\inf(A) = 6$$

$$\sup(A) = 2$$



(93)

upper bound of $B = \text{no}$
 lower bound of $B = \{6, 7, 8\}$.
 $\mu(B)$ does not exist
 there is no upper
 bound of B .



$\mu(B)$ does not exist as
 lower bound succeeds both 6 and 7.

Remark. Every ^{ordered} linearly $\uparrow$ set is a lattice.
 sublattice.

Let M be a non empty subset of a lattice L .
 then M is sublattice of L if M itself is a lattice
 with respect to the operations of L .

For example the set D_m of divisors of m is a
 sublattice of $P = \{1, 2, 3, \dots\}$ under divisibility.

Isomorphic Lattices.

Two lattices L_1 and L_2 are said to be isomorphic
 if there is a one-to-one correspondence $f: L_1 \rightarrow L_2$
 such that

$$f(a \wedge b) = f(a) \wedge f(b)$$

$$\text{and } f(a \vee b) = f(a) \vee f(b).$$

for any elements $a, b \in L_1$.

Bounded Lattice.

A lattice L is said to be bounded below by 0
 if for any $x \in L$ we have $0 \leq x$. Similarly, L is

(94)

said to be bounded above by I if for any $x \in L$
we have $x \leq I$.

A lattice L is bounded if it has upper bound and lower bound.

In bounded lattices the following identities hold.

$$a \wedge I = a \quad ; \quad a \vee I = I$$

$$0 \wedge a = 0 \quad ; \quad 0 \vee a = a$$

Theorem.

Every finite lattice L is bounded.

Proof.

Let $L = \{a_1, a_2, \dots, a_n\}$ be a finite lattice.

Let $x = a_1 \wedge a_2 \wedge \dots \wedge a_n$ and $y = a_1 \vee a_2 \vee \dots \vee a_n$.

Let $a_i \in L$. First we check $x \leq a_i$ as follows

$$\begin{aligned} a_i \vee x &= a_i \vee (a_1 \wedge a_2 \wedge \dots \wedge a_n) \\ &= a_i \vee (a_1 \wedge a_2 \wedge \dots \wedge a_{i-1} \wedge a_i \wedge a_{i+1} \wedge \dots \wedge a_n) \\ &= a_i \vee (a_i \wedge a_1 \wedge a_2 \wedge \dots \wedge a_{i-1} \wedge a_{i+1} \wedge \dots \wedge a_n) \\ &= a_i \vee (a_i \wedge x') \quad \text{where } x' = a_1 \wedge a_2 \wedge \dots \wedge a_{i-1} \wedge a_{i+1} \wedge \dots \wedge a_n \\ &= a_i \quad \text{by absorption law.} \end{aligned}$$

$$\Rightarrow a_i \vee x = a_i$$

$$\Rightarrow x \leq a_i$$

As a_i is an arbitrary element of L ,
for all $a_i \in L$.

$\Rightarrow L$ is bounded below.

i.e. x is lower bound of L .

Hint $x \leq a_i \Leftrightarrow x \wedge a_i = x$
 $a_i \vee x = a_i$

(95)

$a_i \in L$. We check $a_i \leq y$ as follows:

$$\begin{aligned} a_i \wedge y &= a_i \wedge (a_1 \vee a_2 \vee \dots \vee a_n) \\ &= a_i \wedge (a_1 \vee a_2 \vee \dots \vee a_{i-1} \vee a_i \vee a_{i+1} \vee \dots \vee a_n) \\ &= a_i \wedge (a_i \vee a_1 \vee a_2 \vee \dots \vee a_{i-1} \vee a_{i+1} \vee \dots \vee a_n) \\ &= a_i \wedge (a_i \vee y') \end{aligned}$$

where $y' = a_1 \vee a_2 \vee \dots \vee a_{i-1} \vee a_{i+1} \vee \dots \vee a_n$.

$$a_i \wedge y = a_i$$

$$a_i \leq y$$

As a_i is an arbitrary element of L , $a_i \leq y \forall a_i \in L$.

$\Rightarrow y$ is upper bound of L .

$\Rightarrow L$ is bounded above.

Hence every finite lattice L is bounded.

Distributive Lattice.

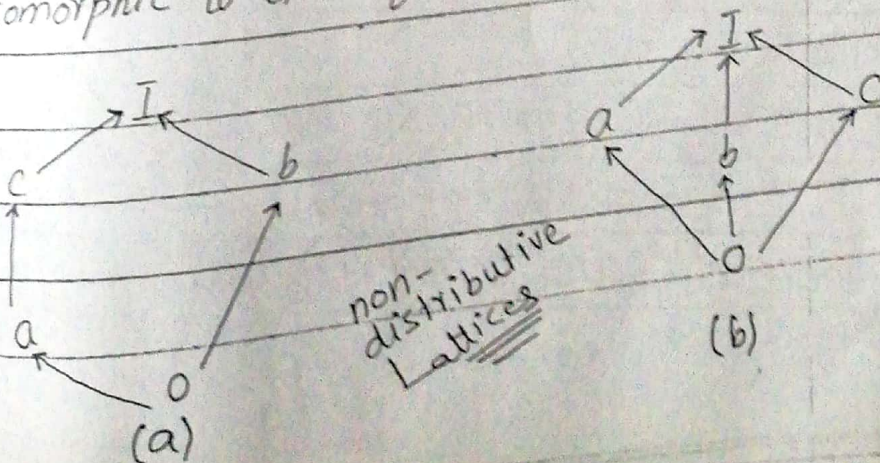
A lattice L is said to be distributive if, for any elements a, b, c in L , the following axioms hold.

$$(i) a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

$$(ii) a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

Remark.

A lattice is non distributive iff it contains a sublattice isomorphic to one of the lattices given as



(96)

Join-Irreducible Element.

Let L be the lattice with a lower bound 0 .
An element $a \in L$ is said to be join-irreducible if it cannot be written as a join of two elements.

i.e. if $a = x \vee y$ then either $a = x$ or $a = y$.

Remarks.

① 0 is always join-irreducible.

② An element $a \in L$ is join-irreducible if and only if 'a' has a unique immediate predecessor.

Atoms.

Those elements which immediately succeed 0 are called atoms.

Atom has unique predecessor so atoms are join-irreducible.

Theorem.

Let L be a finite distributive lattice.

Then every a in L can be written uniquely as the join of irredundant join-irreducible elements.

Proof.

Since L is finite so we have

$$a = d_1 \vee d_2 \vee \dots \vee d_n \longrightarrow \text{①}$$

Where d 's are join-irreducible.

(97)

$d_i \leq d_j$ then $d_i \vee d_j = d_j$. $\therefore a \leq b \Leftrightarrow a \vee b = b$
we can eliminate d_i from the expression.
other words, we can assume that the d 's
are irredundant.

no d precedes any other d .
for uniqueness, suppose

$$a = b_1 \vee b_2 \vee \dots \vee b_n \longrightarrow (2)$$

$$a = c_1 \vee c_2 \vee \dots \vee c_m$$

where b 's and c 's are irredundant and join-irreducible elements.

For any given i we have

$$b_i \leq (b_1 \vee b_2 \vee \dots \vee b_n)$$

$$\Rightarrow b_i \leq (c_1 \vee c_2 \vee \dots \vee c_m) \quad \therefore \text{using (2)}$$

Since L is distributive, therefore,

$$b_i = b_i \wedge (c_1 \vee c_2 \vee \dots \vee c_m) \quad \therefore a \leq b \Leftrightarrow a \wedge b = a$$

$$b_i = (b_i \wedge c_1) \vee (b_i \wedge c_2) \vee \dots \vee (b_i \wedge c_m)$$

Since b_i is join-irreducible, there exists a j
such that

$$b_i = b_i \wedge c_j \longrightarrow (3)$$

$$\Rightarrow b_i \leq c_j$$

By similar argument, for c_j there exist b_k

such that

$$c_j \leq b_k \longrightarrow (4)$$

From (3) and (4), we can write,

$$b_i \leq c_j \leq b_k \longrightarrow (5)$$

(98)

$$\Rightarrow b_i \leq b_k$$

Since b 's are irredundant so (6) is only possible when

$$b_i = b_k$$

From (5), $b_i = c_j = b_k$

Accordingly, the b 's and c 's may be paired off. Thus the representation for a is unique except for order.

Complement of an Element.

Let L be a bounded lattice with lower bound 0 and upper bound I . Let ' a ' be an element of L . An element x in L is called a complement of a if

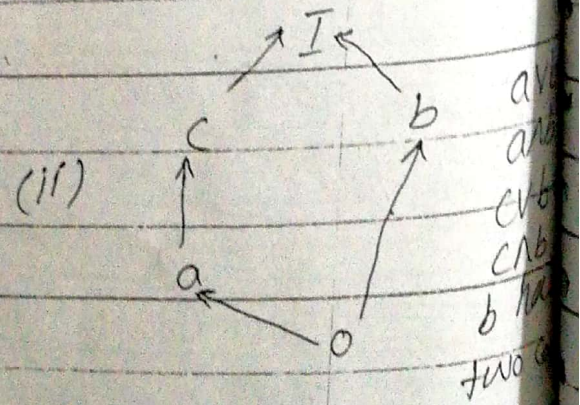
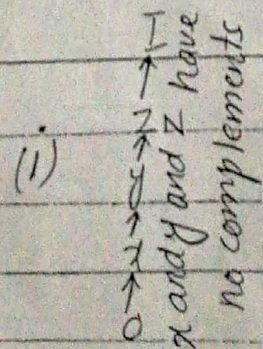
$$a \wedge x = 0 \text{ and } a \vee x = I$$

Remarks.

- (i) In general the complements may or may not exist.
- (ii) If complement exists, it may or may not be unique.

For example.

(Counter examples)



(99)

Complemented Lattices

A lattice L is said to be Complemented if L is bounded and every element in L has a Complement.

For example, the lattice $P(U)$ of all subsets of a universal set U is Complemented and each subset A of U has the unique complement $A^c = U \setminus A$.

Theorem.

Let L be a bounded distributive lattice. Then Complements are unique if they exist.

Proof.

Suppose $x \in L$ has two Complements, say, a and b .

$$\text{Then } x \wedge a = 0 \quad ; \quad x \wedge b = 0 \quad \longrightarrow \textcircled{1}$$

$$\text{and } x \vee a = I \quad ; \quad x \vee b = I$$

Since 0 preceeds every number, therefore,

$$a = a \vee 0$$

$$= a \vee (x \wedge b) \quad \because \text{using } \textcircled{1}$$

$$= (a \vee x) \wedge (a \vee b) \quad \because L \text{ is distributive.}$$

$$= I \wedge (a \vee b) \quad \because \text{using } \textcircled{1}.$$

$$\Rightarrow a = a \vee b \quad \text{---} \textcircled{2}$$

$$\text{Similarly, } b = b \vee 0$$

$$= b \vee (x \wedge a) \quad \because \text{using } \textcircled{1}$$

$$= (b \vee x) \wedge (b \vee a) \quad \because L \text{ is distributive.}$$

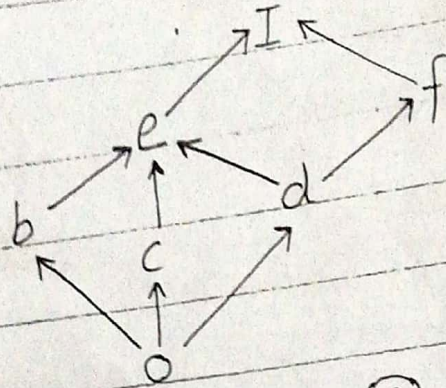
$$= I \wedge (b \vee a) \quad \text{using } \textcircled{1}$$

$$= b \vee a$$

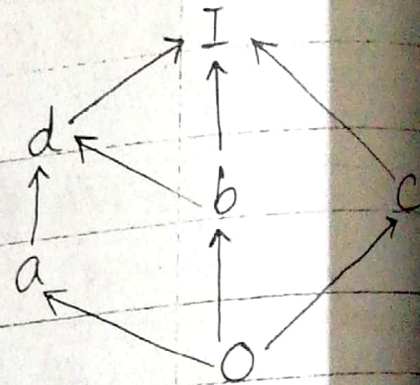
Exercise.

Given the lattices

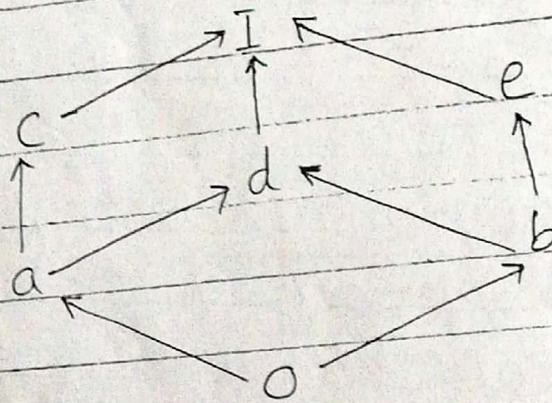
(a)



(b)



(c)



- ① Find the complement of
(i) e and f in Fig (a) (ii) a and c in Fig (b)
- ② Express I in an irredundant join irreducible decomposition in as many ways as possible for fig (a) and (b).
- ③ Are the lattices given in (a), (b) and (c) distributive or not?
- ④ Describe the isomorphism of lattice in (a) and (c) with itself.

Solution.

(103)

(1)

i) The complement of e does not exist and of f are b and c in Fig(a).

ii) The complement of a is c and c has b and a in Fig(b).

(2)

For Fig(a)

$$I = c \vee d \vee f = b \vee c \vee f = b \vee d \vee f$$

For Fig(b).

$$I = a \vee c = b \vee c$$

For Fig(c).

$$I = a \vee e = b \vee c = c \vee e$$

(3)

Fig(a): This lattice is not distributive.

Fig(b): This lattice is not distributive.

Fig(c): This lattice is no distributive.

(4)

p 3:

Ordinal Numbers

Well-Ordered Set. 2012,

Let A be an ordered set. Then A is said to be well-ordered if every subset of A has a first element.

e.g. the set of natural numbers is well-ordered.

Let A be a well-ordered set. Then

(i) Every subset of A is well-ordered.

(ii) If B is similar to A , then B is well ordered.

Union order / Concatenation order / Usual order:

Let $\{A_i : i \in I\}$ be a linearly ordered collection of disjoint linearly ordered sets, that is, I is linearly ordered and each A_i is linearly ordered.

Then the union $S = \cup A_i$ will be linearly ordered as follows:

$$a < b \quad \text{if} \quad \begin{cases} a \in A_i, b \in A_j, i < j \\ a, b \in A_i, a < b \text{ in } A_i. \end{cases}$$

This ordering is called the usual ordering on the union S .

For example,

Let $A = \{1, 3, 5, 7, \dots\}$, $B = \{a, b, c\}$, $C = \{2, 4, 6, \dots\}$

Then concatenation on $S = A \cup B \cup C$, ($A < B < C$)

is $S = \{1, 3, 5, \dots; a, b, c; 2, 4, 6, \dots\}$.

Remarks.

① Every well ordered set is linearly ordered because if A is well ordered and $a, b \in A$

(106)

then $\{a, b\}$ has 1st element which precedes other. Hence a and b are comparable.
(2) Linearly ordered set may not be well ordered in general.

For example the set of integers $\{0, \pm 1, \pm 2, \dots\}$ is linearly ordered but $\{\dots, -3, -2, -1\} \subseteq \{0, \pm 1, \pm 2, \dots\}$ has no 1st element. Therefore the set of integers is not well ordered.

Theorem.

Suppose $\{A_i : i \in I\}$ is a well-ordered family of disjoint well ordered sets, that is, I is well ordered and each A_i is well ordered. Then the union $S = \bigcup A_i$, with the usual ordering, is well ordered.

Proof.

Let B be any subset of $S = \bigcup A_i$. Then for any $b \in B$, $b \in A_i$ for some A_i .

The usual order on the union $S = \bigcup A_i$ is defined as

$$a < b \text{ if } \begin{cases} a \in A_i, b \in A_j, i < j \\ a, b \in A_i, a < b \text{ in } A_i \end{cases}$$

Let j be the smallest index such that

$$A_j \cap B \neq \emptyset$$

As A_j is well-ordered so $A_j \cap B$ has element ' a ' say.

Then a is also first element of B in usual order.

Theorem.

All finite linearly ordered sets with the same number n of elements are well-ordered and are similar to each other.

Proof.

Let $V = \{a_1, a_2, \dots, a_n\}$ be any finite linearly ordered set. Then V can be written as linear chain $V = \{a_{i_1}, a_{i_2}, \dots, a_{i_n}\}$ where the elements are ordered as shown (linear chain).

Since any subset of linear chain is also linear chain and V is finite.

Therefore, each subset of V has 1st element.

Hence V is well ordered.

Now, let $W = \{b_1, b_2, \dots, b_n\}$ be any other linearly ordered set with n elements. Then W can also be written as linear chain

$W = \{b_{i_1}, b_{i_2}, \dots, b_{i_n}\}$ where the elements are ordered as shown (linear chain).

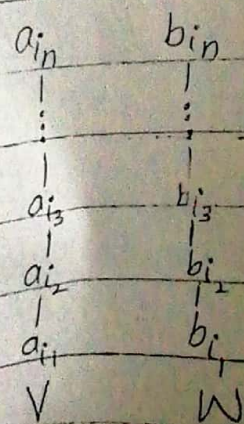
Let $f: V \rightarrow W$ be a mapping defined as

$$f(a_{i_j}) = b_{i_j}$$

Then f is clearly an order preserving bijective mapping.

i.e. f is similarity mapping.

$$V \approx W.$$



2015
Statement and prove principle of transfinite induction.

Statement:

Let S be a subset of a well-ordered set A with the following two properties:

- (i) $a_0 \in S$ ^{where a_0 is first element} (ii) if $s(a) \subseteq S$, then $a \in S$.

Then $S = A$.

Proof.

Let $S \subseteq A$ satisfying properties (i) and (ii). We have to show $S = A$. Suppose on contrary, $S \neq A$.

$$\Rightarrow T = A \setminus S \neq \emptyset.$$

Then T is a subset of well-ordered set A .

This further shows that T has first element say t_0 .

Consider $s(t_0)$. Then for each $x \in s(t_0)$ $x < t_0$.

Then $x \notin T$ because t_0 is first element of T .

$$\Rightarrow x \in S \quad \forall x \in s(t_0).$$

$$\Rightarrow s(t_0) \subseteq S$$

By property (ii) $t_0 \in S$.

which is a contradiction as $t_0 \in T = A \setminus S$.

Hence $S = A$.

Theorem A/2018, 2013, 2014,

Every element in a well-ordered set A has a unique immediate successor except the last element. ^{یعنی}

(109)

Proof.

Let $a \in A$ and $M(a)$ be the set of all elements of A which strictly succeeds a .

$$\text{i.e. } M(a) = \{x : a < x\}.$$

Since a is not last element, $M(a) \neq \emptyset$.

Then $M(a)$ has first element say b being the subset of a well-ordered set. We claim that b is immediate successor of a . For if b is not immediate successor, then there exists $c \in A$ such that

$$a < c < b.$$

$$\Rightarrow c \in M(a)$$

which is contradiction to the assumption that b is the 1st element of $M(a)$.

Hence b is immediate successor of a .

We claim that b is the unique immediate successor of a . For if there is an other immediate successor d of a then $a < b$ and $a < d$.

Then $b, d \in M(a)$.

Since b is the 1st element so $b < d$.

$$\Rightarrow a < b < d$$

$\Rightarrow d$ is not immediate successor which is a contradiction.

Hence b is the only immediate successor of a .

5/2018

(110)

Remark.

The above statement may not be true for immediate predecessor of elements of well-ordered set.

For example in the set $P = \{1, 3, 5, \dots, 2, 4, 6, \dots\}$ with sum order. In this case 1 and 2 have no immediate predecessor.

Limit element.

An element "a" in a well-ordered set A is called a limit element if it does not have an immediate predecessor and if it is not the first element.

For example $S = \{1, 3, 5, \dots, 2, 4, 6, \dots\}$.

Here 1 and 2 do not have immediate predecessor.

Theorem.

Let $SC(A)$ denote the collection of all initial segments of elements in a well-ordered set A, and let $SC(A)$ be ordered by set inclusion. Then A is similar to $SC(A)$.

i.e. $A \approx SC(A)$.

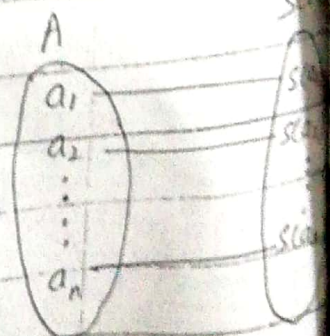
Proof.

Define a mapping

$f: A \rightarrow SC(A)$ as

$$f(a) = s(a) = \{x: x < a\}.$$

f is onto.



(III)

For each $s(a) \in S(A)$ there exists $a \in A$ such that

$$f(a) = s(a).$$

This shows that f is onto.

f is 1-1.

For this let $a_1, a_2 \in A$ such that

$$a_1 \neq a_2.$$

Since A is well-ordered, so one of them strictly precedes the other.

$$\text{Let } a_1 < a_2. \Rightarrow a_1 \in s(a_2)$$

$$\text{Also } a_1 \notin s(a_1).$$

$$\Rightarrow s(a_1) \neq s(a_2)$$

$$\Rightarrow f(a_1) \neq f(a_2).$$

$$\Rightarrow f \text{ is one-one.}$$

$$\Rightarrow f \text{ is bijective mapping.}$$

Now we show that f preserves order.

$$\text{i.e. } a \leq b \Leftrightarrow s(a) \subseteq s(b).$$

$\left\{ \begin{array}{l} \text{order given} \\ \text{set inclusion} \end{array} \right.$

Suppose $a \leq b$. let $x \in s(a)$. Then $x < a$.

$$\Rightarrow x < a \leq b$$

$$\Rightarrow x < b$$

$$\Rightarrow x \in s(b)$$

$$\Rightarrow s(a) \subseteq s(b).$$

Conversely suppose $a \not\leq b$.

Since A is well ordered.

Therefore the only possibility is

$\left\{ \begin{array}{l} p \Rightarrow q \\ \sim q \Rightarrow \sim p \\ \sim \text{not} \end{array} \right.$

(112)

$$a > b$$

$$\text{or } b < a$$

$$\Rightarrow b \in s(a)$$

$$\text{But } b \notin s(b)$$

$$\Rightarrow s(a) \not\subseteq s(b)$$

$\Rightarrow f$ preserves order.

Hence $A \approx s(A)$.

Initial Segment.

Let A be a well-ordered set. The initial segment $s(a)$ of an element $a \in A$ consists of all elements in A which strictly precedes a .

In other words,

$$s(a) = \{x : x \in A, x < a\}$$

Here $s(a)$ is a subset of A .

Theorem. 2018, 2013, 2016,

Let A be a well-ordered set, let B be a subset of A , and let the function $f: A \rightarrow B$ be a similarity mapping of A onto B . Then, for every $a \in A$,

$$a \leq f(a).$$

Proof.

Let $D = \{x \in A : f(x) < x\}$. If $D = \emptyset$, then result holds. We may suppose $D \neq \emptyset$.

Since A is well-ordered so D has 1st element. Say d_0 .

(113)

Then $f(d_0) < d_0$ ——— ①

is $f: A \rightarrow B$ is onto similarity mapping so

$$f(f(d_0)) < f(d_0)$$

let $d' = f(d_0)$. Then $f(d') < d'$
 $\Rightarrow d' \in D$.

Since d_0 is the first element of D .

therefore $d_0 \leq d'$

$$\Rightarrow d_0 \leq f(d_0) \text{ ——— ②}$$

① and ② contradict each other.

Hence $D = \emptyset$.

i.e. $\forall a \in A, a \leq f(a)$.

Theorem. 2011, 2012, 2013,

Let A and B be similar well-ordered sets.

Then there exists only one similarity mapping of A into B .

Proof.

Let $f: A \rightarrow B$ and $g: A \rightarrow B$ be two similar mappings of A and B .

Suppose $f \neq g$. Then there exists $x \in A$ such that $f(x) \neq g(x)$.

Since $f(x), g(x) \in B$ and B is well-ordered so

either $f(x) < g(x)$ or $g(x) < f(x)$.

Without any loss of generality, we can suppose that $f(x) < g(x)$ ——— ①

(114)

Since $g: A \rightarrow B$ is similarity mapping so

$g^{-1}: B \rightarrow A$ is also a similarity mapping.

Also $g \circ f: A \rightarrow A$ is a similarity mapping

As $f(x), g(x) \in B$ so from (1)

$$g^{-1}(f(x)) < g^{-1}(g(x))$$

$$\Rightarrow g \circ f(x) < g \circ g(x)$$

$$\Rightarrow g \circ f(x) < I(x)$$

$$\Rightarrow g \circ f(x) < x$$

image precede

which is a contradiction

as we know that if $f: A \rightarrow B$, $B \subseteq A$, is similarity mapping then $a \leq f(a)$, $\forall a \in A$.

Hence $f = g$.

Theorem. A12012

A well-ordered set cannot be similar to one of its initial segments.

Proof.

Let A be any well-ordered set and $S(a)$ be one of its initial segment.

Suppose on contrary, there exists a similarity mapping $f: A \rightarrow S(a)$.

$$\Rightarrow \text{for } a \in A, f(a) \in S(a)$$

$$\Rightarrow f(a) < a$$

which is a contradiction as we know

that if $f: A \rightarrow S(a)$ is similarity mapping

then $a \leq f(a)$, $\forall a \in A$.

(115)

Theorem. 2016, 2017.

Let A be a well-ordered set and let S be a subset of A with the following property:

If $a \leq b$ and $b \in S$, then $a \in S$.

Then $S = A$ or S is an initial segment of A .

Proof.

If $S = A$. Then we are nothing to show.

Suppose $S \neq A$. We have to show S is an initial segment of A . i.e. $S = \beta(a)$ for some $a \in A$.

Now $S \neq A$, so $T = A \setminus S \neq \emptyset$.

Since A is well-ordered so T has first element say a_0 such that $a_0 \notin S$.

We will show that $S = \beta(a_0)$. For this, let

$$x \in \beta(a_0)$$

$$\text{Then } x < a_0$$

$$\Rightarrow x \notin T \text{ because } a_0 \text{ is 1st element of } T.$$

$$\Rightarrow x \in S.$$

$$\Rightarrow \beta(a_0) \subseteq S \quad \text{--- (1)}$$

To show $S \subseteq \beta(a_0)$, it is sufficient to show that for $y \notin \beta(a_0)$, $y \notin S$.

For this let $y \notin \beta(a_0)$. Then $a_0 \leq y$

On contrary suppose $y \in S$. Then $a_0 \leq y$ and $y \in S$

Then by given property in statement

$$a_0 \in S.$$

Which is a contradiction to our assumption

that $a_0 \notin S$.

$$a \in A \Rightarrow s(a) \subseteq A$$

$$a \in s(b) \Rightarrow s(a) \subseteq s(b)$$

(116)

Hence yes leads a contradiction which shows that $y \notin S$.

$$\Rightarrow S \subseteq s(a_0) \quad \text{--- (2)}$$

From (1) and (2), we have

$$S = s(a_0)$$

Theorem.

Two initial segments of a well-ordered set cannot be similar.

Proof.

For $a \neq b$, let $s(a)$ and $s(b)$ be two different initial segments of a well-ordered set.

Since $a \neq b$, therefore, $a < b$ or $b < a$.

Without any loss of generality, we assume that $a < b$.

Then $a \in s(b)$.

$$\Rightarrow s(a) \subseteq s(b).$$

i.e. $s(a)$ is an initial segment of well-ordered set $s(b)$.

Since we know that a well-ordered set cannot be similar to one of its initial segments.

Therefore, $s(b)$ is not similar to $s(a)$.

Theorem.

Let A and B be well-ordered sets, and

(117)

an initial segment $s(a)$ of A be similar to an initial segment of B . Then $s(a)$ is similar to a unique initial segment $s(b)$ of B .

Proof.

Let $s(a) \simeq s(b)$ and $s(a) \simeq s(b')$ where $b, b' \in B$.

Then $s(b) \simeq s(b')$.

As we know that two initial segments of a well-ordered set cannot be similar.

So only possibility is that

$$s(b) = s(b').$$

Hence $s(a)$ is similar to a unique initial segment $s(b)$ of B .

Definition.

Let A and B be two well-ordered sets. Then A is said to be shorter than B if A is similar to initial segment of B .

In this case B is said to be longer than A .

Theorem. 2013, 2014,

Let A and B be two well-ordered sets.

Then either A and B are similar or one of them is similar to initial segment of other.

OR

Let A and B be two well-ordered sets.

Then A is shorter than B or A is similar to B or A is longer than B .

Proof.

Consider $S = \{x \in A : \exists y \in B \text{ such that } s(x) \approx s(y)\}$
 and $T = \{y \in B : \exists x \in A \text{ such that } s(y) \approx s(x)\}$
 First we show that $S = A$ or S is an initial segment of A .

Let $x \in S$ and $z \leq x$. Then by definition of initial segment $z \in S(x)$ and $s(x) \approx s(y)$ (given).

$\Rightarrow$ there exists z' such that $s(z) \approx s(z') \subseteq s(y) \subseteq B$.

$\Rightarrow s(z)$ is similar to initial segment of B .

$\Rightarrow z \in S$.

Since we know that if a set X of an ordered set A satisfies $a \leq b, b \in X \Rightarrow a \in X$.

then either $X = A$ or X is an initial segment of A .

Therefore $S = A$ or S is an initial segment of A .

Similarly, we can show that $T = B$ or T is an initial segment of B .

Now we will show that $S \approx T$. For this let $x \in S$. Then as we know that $s(x)$ is similar to a unique segment $s(y)$ of B .

Thus to each $x \in S$, there corresponds a unique $y \in T$ such that $s(x) \approx s(y)$ and vice versa.

Hence the function $f : S \rightarrow T$ defined by

$$f(x) = y$$

(119)

is 1-1 and onto. $\therefore$ if $s(x) \approx s(y)$.

To show f preserves order we let $x, x' \in S$

such that

$f(x) = y$, $f(x') = y'$ and $x' < x$.

We have to show that $y' < y$.

Since $s(x) \approx s(y)$

there exists a similarity mapping

$g: s(x) \rightarrow s(y)$

of $s(x)$ into $s(y)$. Then g is restricted to $s(x')$ is the similarity mapping of $s(x')$ to $s(g(x'))$.

i.e. $s(x') \approx s(g(x'))$.

But $f(x') = y'$.

So $s(x') \approx s(y')$.

Since every initial segment of S is similar to unique initial segment of T .

Therefore $s(g(x')) \approx s(y')$

Consequently,

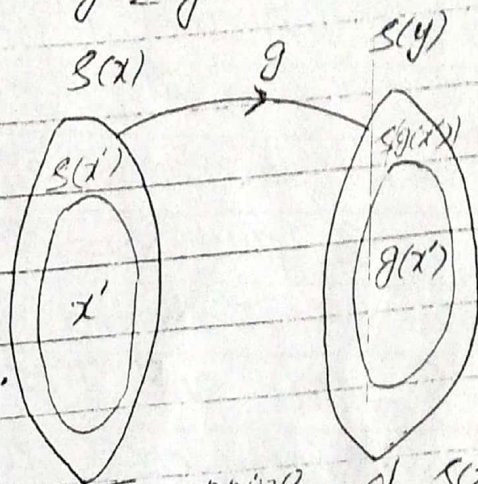
$g(x') = y'$.

Since $g(x') \in s(y)$.

Therefore $g(x') < y$

$\Rightarrow y' < y \quad \therefore g(x') = y'$

$\Rightarrow f(x') < f(x)$



(120)

This shows that $x' < x \Rightarrow f(x') < f(x)$.
 $\Rightarrow f$ preserves order.

Hence $S \approx T$.

There arise four cases.

- (1) $S=A$ and $T=B \Rightarrow A \approx B$.
- (2) $S=A$ and $T=f(b) \Rightarrow A \approx f(b)$.
 $\Rightarrow A$ is shorter than B .
- (3) $T=B$ and $S=f(a) \Rightarrow B \approx f(a)$
i.e. B is shorter than A .
- (4) $S=f(a)$ and $T=f(b) \Rightarrow f(a) \approx f(b)$.

Then clearly $a \in S$ as $f(a)$ is similar to initial segment of B .

$$\Rightarrow a \in S = f(a)$$

$$\Rightarrow a \in f(a)$$

which is a contradiction as no element belongs to its own initial segment.

Thus Case (4) is not possible.

From case (1), (2) and (3) the theorem is proved.

Theorem.

Let A be a collection of initial segments of a well-ordered set A . Then there is an initial segment $f(a) \in A$ such that $f(a) \subseteq$ for any other initial segment $f(x)$ in A , i.e. there is an initial segment $f(a) \in A$ which is shorter than every other initial segment in A .

Proof.

Let A be a well-ordered set ^{and} $S(A)$ denote the family of all initial segments in A . Then there is a mapping $f: A \rightarrow S(A)$ which is one-one, onto and order preserving.

i.e. $A \approx S(A)$

Since A is well-ordered, $S(A)$ is also well-ordered.

Since A is a subset of $S(A)$ so A has a first element $s(a)$. Therefore $s(a) \subseteq s(x)$ for any other initial segment in A .

Theorem.   - استعمال پر ہے۔ Theorem 3.4 

Let A be a collection of pairwise non-similar well-ordered sets. Then there exists a set A_0 in A such that A_0 is shorter than every other set in A .

Proof.

Let A be a collection of pairwise non-similar well-ordered sets. Let B be any subset of A .

Now we define a family of sets

$$B = \{x : x \in A, x \text{ is shorter than } B\}.$$

Case I.

If $B = \emptyset$ then B satisfies the condition that it is shorter than every set in A and theorem is proved.

Case II:

If $B \neq \emptyset$, then we show that B has a shortest set A_0 (say) which will be shorter than every other set in A .

Since B contains the sets which are shorter than set B , therefore, for any set $A \in B$, A is similar to an initial segment $s(a)$ of set B .

Let B' be the collection of all those initial segments of B each of which is similar to a set in B .

Since for a collection of initial segments of a well-ordered set there is an initial segment $s(a)$ in collection which is shorter

than every other initial segment in the collection.

Therefore B' contains an initial segment $s(a_0)$ which is shorter than every other initial segment in B' .

Consequently, the set $A_0 \in B$ which is similar to $s(a_0)$ is shorter than every other set in B . Therefore A_0 is shorter than every other set in A .

This completes the proof.

Ordinal Numbers. 2011, 2017,

Let $\mathcal{L}$ be the collection of all well-ordered sets. Every well-ordered set A in $\mathcal{L}$ is assigned a symbol in such a way that any two well-ordered sets A and B are assigned the same symbol if and only if A and B are similar. This symbol is called the ordinal number of A and is written as

$$\lambda = \text{ord}(A).$$

Remarks.

For any two well-ordered sets A and B of same cardinality n .

$$\text{ord}(A) = \text{ord}(B) = n$$

So up to isomorphism there is only well-ordered set for ordinal number n .

② If A is finite well-ordered set having n elements then $\text{ord}(A) = n$

For example $A = \{1\} \rightarrow \text{ord}(A) = 1$

$$B = \{1, 2\} \rightarrow \text{ord}(B) = 2$$

Transfinite Ordinal numbers.

Infinite ordinal numbers are called transfinite ordinal numbers.

Note:

The ordinal number of set $P = \{1, 2, 3, \dots\}$ is

denoted by $\text{ord}(P) = \omega$, $\text{ord}(Z)$ by $\aleph$ and $\text{ord}(Q)$ by $\aleph_1$.

(124)

Inequalities for ordinal numbers.

Let λ and μ be two ordinal numbers and let A and B be two well-ordered sets such that $\lambda = \text{ord}(A)$ and $\mu = \text{ord}(B)$.

Then $\lambda < \mu$ if A is similar to an initial segment of B .

Similarly, we can write

$\lambda < \mu$ if A is shorter than B .

$\lambda > \mu$ if A is longer than B .

$\lambda = \mu$ if A is similar to B .

$\lambda \leq \mu$ if $\lambda < \mu$ or $\lambda = \mu$.

$\lambda \geq \mu$ if $\lambda > \mu$ or $\lambda = \mu$.

Example.

Let $P = \{1, 2, 3, \dots\}$ with prescribed order such that $\text{ord}(P) = \omega$ and if we consider

$P = \{1, 2, 3, \dots; 2, 4, 6, \dots\}$ with prescribed order such that $\text{ord}(P) = \lambda$.

~~Let~~ Then $S(2) = \{1, 2, 3, \dots\}$ in 2nd case

So $P \approx S(2)$.

$\Rightarrow \omega < \lambda$

Similarly we can show

$\omega < \bar{\pi}, \omega < \eta$

(125)

3. Theorem.

Any set of ordinal numbers is linearly ordered by the relation $\lambda \leq \mu$.

Proof.

Let $\mathcal{L}$ be the set of ordinal numbers and let $\lambda, \mu \in \mathcal{L}$. Then there exist two well-ordered sets A and B such that

$$\lambda = \text{ord}(A), \mu = \text{ord}(B).$$

Since we know that for any two well-ordered sets, either A is ~~ord~~ isomorphic to B , or A is shorter than B , or A is longer than B .

i.e. $A \approx B$ or $A \approx s(b)$; $s(b) \subseteq B$
or $B \approx s(a)$; $s(a) \subseteq A$.

$\Rightarrow$ either $\lambda = \mu$ or $\lambda < \mu$ or $\lambda > \mu$.

$\Rightarrow \mathcal{L}$ is linearly ordered.

Hence any set of ordinal numbers is linearly ordered by the relation $\lambda \leq \mu$.

Theorem 1/2018, 2011,

Let $s(\lambda)$ be the initial segment of ordinal numbers λ . Then $\text{Ord}(s(\lambda)) = \lambda$.

Proof.

Let A be the well-ordered set such that $\lambda = \text{ord}(A)$.

Let $S(A)$ be the collection of all initial segments

of A ordered by set inclusion.

i.e. $S(A) = \{s(a) : \forall a \in A\}$ and $s(a) \leq s(a') \Leftrightarrow s(a) \subseteq s(a')$

Then $f: A \rightarrow S(A)$ defined by $f(a) = s(a)$ is an ordered isomorphism similarity mapping.

$$\Rightarrow A \approx S(A)$$

$$\Rightarrow \text{ord}(A) = \text{ord}(S(A))$$

$$\Rightarrow \lambda = \text{ord}(S(A))$$

ord($s(\lambda)$)
sim

(h)

To prove $\lambda = \text{ord}(s(\lambda))$, it is sufficient to prove $S(A) \approx s(\lambda)$.

Define a function $f: s(\lambda) \rightarrow S(A)$ as,
for $\mu \in s(\lambda)$

$$f(\mu) = s(a), \text{ where } \mu = \text{ord}(s(a))$$

f is 1-1.

P.u

id $\mu, \lambda \in s(\lambda)$ are similar
 $\mu + 1 \rightarrow s(\mu) \neq s(\lambda)$

Since $\mu \in s(\lambda)$, so $\mu < \lambda$.

Let $\mu = \text{ord}(B)$. Then B is shorter than
i.e. $B \approx s(a)$ for some $a \in A$.

$$\Rightarrow \mu = \text{ord}(s(a))$$

(1)

We claim that $s(a)$ is the unique such in
segment for if there exists $s(a')$ such that

$$\mu = \text{ord}(s(a'))$$

(2)

From (1) and (2), we have,

$$\text{ord}(s(a)) = \text{ord}(s(a'))$$

$$\Rightarrow s(a) \approx s(a')$$

which contradicts the fact that the

(127)

different initial segments of A cannot be similar.

Hence $s(a)$ is the unique initial segment of A such that $\mu = \text{ord}(s(a))$

$\Rightarrow f$ is one-one.

f is onto.

For this let $s(b) \in S(A)$. Then $s(b)$ is shorter than A and therefore $\text{ord}(s(b)) < \text{ord}(A)$

(say) $\nu = \text{ord}(s(b))$

$\Rightarrow \nu < \lambda$

$\because \text{ord}(A) = \lambda$

$\Rightarrow \nu \in S(\lambda)$

By definition

$f(\nu) = s(b)$ where $\nu = \text{ord}(s(b))$

$\Rightarrow f$ is onto.

Now we show that f preserves order.

For this let $\mu, \nu \in S(\lambda)$ such that

$\mu < \nu$

Then $\mu = \text{ord}(s(a))$ and $\nu = \text{ord}(s(b))$

ie. $f(\mu) = s(a)$ and $f(\nu) = s(b)$

Since $\mu < \nu$, $s(a)$ is ordered isomorphic to some initial segment of $s(b)$ say $s(b')$.

$\Rightarrow s(a) \cong s(b') \subseteq s(b)$

$\Rightarrow f(\mu) < f(\nu)$

where $\mu = \text{ord}(s(a)) = \text{ord}(s(b'))$

$\Rightarrow f$ preserves order

$s(a)$ is an initial segment of $s(b)$.

Hence $s(a)$ is proper subset of $s(b)$

$\Rightarrow s(a) \in s(b)$

Hence $S(A) \approx S(\lambda)$ ————— (3)

Consequently, from (A) and (3), we have

$$\text{ord}(S(A)) = \text{ord}(S(\lambda))$$

$$\Rightarrow \lambda = \text{ord}(S(\lambda)) \quad \text{as required.}$$

Addition of ordinal numbers.

Let λ and μ be two ordinal numbers, and let A and B be two disjoint well-ordered sets such that

$$\lambda = \text{ord}(A) \quad \text{and} \quad \mu = \text{ord}(B).$$

$$\text{Then} \quad \lambda + \mu = \text{ord}(\{A; B\}).$$

Where $\{A; B\}$ is given the usual order.

Remarks.

① Addition of two ordinal numbers is not commutative in general.

For example

$$\text{Let } P = \{1, 2, 3, \dots\} \quad \text{and} \quad A = \{a, b\}$$

$$\text{Also let } \text{ord}(P) = \omega \quad \text{and} \quad \text{ord}(A) = 2.$$

$$\text{Then } \omega + 2 = \text{ord}(\{P; A\})$$

$$= \text{ord}(\{1, 2, 3, \dots; a, b\})$$

$$\text{Since } S(a) = \{1, 2, 3, \dots\}$$

$$\text{i.e. } S(a) \approx P$$

$$\Rightarrow \omega + 2 > \text{ord}(P) = \omega$$

$$\Rightarrow \omega + 2 > \omega \quad \text{————— (1)}$$

$$\text{Now } 2 + \omega = \text{ord}(\{A; P\})$$

$$= \text{ord}(\{a, b; 1, 2, 3, \dots\}) = \omega$$

$$\text{as } \{a, b; 1, 2, 3, \dots\} \approx \{1, 2, 3, \dots\}$$

(129)

$$\Rightarrow 2 + \omega = \omega$$

(2)

From (1) and (2), we have

$$\omega + 2 \neq 2 + \omega$$

(2) In case of finite ordinal numbers addition of ordinal numbers is commutative.

Theorem. 2012, 2013, 2017,

Let λ be any ordinal number. Then $\lambda + 1$ is the immediate successor of λ .

Proof.

Let μ be the immediate successor of λ .

Then by definition of initial segments

$$s(\mu) = \{x \mid x < \mu\} \text{ and } s(\lambda) = \{y \mid y < \lambda\}.$$

Since $\lambda < \mu$ so we can write,

$$s(\mu) = s(\lambda) \cup \{\lambda\}$$

$$\Rightarrow \text{ord}(s(\mu)) = \text{ord}(s(\lambda); \lambda)$$

$$\Rightarrow \mu = \text{ord}(s(\lambda)) + \text{ord}(\{\lambda\})$$

$$\Rightarrow \mu = \lambda + 1.$$

Theorem.

For any ordinal numbers λ, μ, η .

$$(i) (\lambda + \mu) + \eta = \lambda + (\mu + \eta)$$

$$(ii) \lambda + 0 = 0 + \lambda.$$

Proof.

(i) Let $A = \{a_1, a_2, \dots\}$, $B = \{b_1, b_2, \dots\}$ and

$C = \{c_1, c_2, \dots\}$ such that

$$\text{ord}(A) = \lambda, \text{ord}(B) = \mu \text{ and } \text{ord}(C) = \eta$$

$$\begin{aligned}
 \text{Then } (\lambda + \mu) + \eta &= \text{ord}(\{ \{a_1, a_2, \dots\} b_1, b_2, \dots \}; c_1, c_2, \dots) \\
 &= \text{ord}(\{ a_1, a_2, \dots; b_1, b_2, \dots; c_1, c_2, \dots \}) \\
 &= \text{ord}(\{ a_1, a_2, \dots; (b_1, b_2, \dots; c_1, c_2, \dots) \}) \\
 &= \lambda + (\mu + \eta) \\
 &\quad (ii')
 \end{aligned}$$

Let $A = \{a_1, a_2, \dots\}$ and $D = \emptyset$ such that
 $\text{ord}(A) = \lambda$ and $\text{ord}(D) = 0$

$$\begin{aligned}
 \text{Then } \lambda + 0 &= \text{ord}(\{a_1, a_2, \dots; 0\}) \\
 &= \text{ord}(\{a_1, a_2, \dots\})
 \end{aligned}$$

$$\Rightarrow \lambda + 0 = \lambda \quad \text{--- (1)}$$

$$\begin{aligned}
 \text{and } 0 + \lambda &= \text{ord}(\{0; a_1, a_2, \dots\}) \\
 &= \text{ord}(\{a_1, a_2, \dots\}) \\
 &= \text{ord}(A)
 \end{aligned}$$

$$\Rightarrow 0 + \lambda = \lambda \quad \text{--- (2)}$$

By (1) and (2), we have

$$0 + \lambda = \lambda + 0$$

General Addition:

Let $\{\lambda_i : i \in I\}$ be a well ordered collection of ordinal numbers such that $\lambda_i = \text{ord}(A_i)$

Then

$$\sum_{i \in I} \lambda_i = \text{ord}\left(\bigcup_{i \in I} \{A_i \times \{i\}\}\right)$$

Multiplication of ordinal numbers. 2018

Let λ and μ be two ordinal numbers and let A and B be two well-ordered

(131)

sets such that:

$$\lambda = \text{ord}(A) \text{ and } \mu = \text{ord}(B)$$

$$\text{Then } \lambda\mu = \text{ord}(A \times B)$$

where $A \times B$ is ordered reverse lexicographically.i.e. for $(a, b), (c, d) \in A \times B$

$$(a, b) < (c, d) \text{ iff } \begin{cases} b < d \\ \text{or } b = d \text{ and } a < c \end{cases}$$

Remark.

Multiplication of two ordinal numbers is not commutative in general.

For example

$$\omega^2 \neq \omega \cdot \omega$$

Let $A = \{1, 2, 3, \dots\}$ and $B = \{a, b\}$ such that $\text{ord}(A) = \omega$ and $\text{ord}(B) = 2$

$$\text{Then } A \times B = \{(1, a), (2, a), (3, a), \dots; (1, b), (2, b), (3, b), \dots\}$$

$$\text{and } B \times A = \{(a, 1), (b, 1), (a, 2), (b, 2), (a, 3), (b, 3), \dots\}$$

So

$$\omega^2 = \text{ord}(A \times B)$$

$$= \text{ord}(\{(1, a), (2, a), (3, a), \dots; (1, b), (2, b), (3, b), \dots\})$$

$$\text{Since } \mathcal{S}((1, b)) = \{(1, a), (2, a), (3, a), \dots\} \approx A$$

$$\Rightarrow \mathcal{S}((1, b)) \approx A$$

$$\Rightarrow \omega^2 \supseteq \text{ord}(A) = \omega$$

$$\Rightarrow \omega^2 > \omega \quad \text{--- (1)}$$

$$\text{and } 2\omega = \text{ord}(B \times A)$$

$$= \text{ord}(\{(a, 1), (b, 1), (a, 2), (b, 2), \dots\})$$

Since ~~the set~~ ~~is~~ ~~infinite~~ ~~...~~

$\{(a, 1), (b, 1), (a, 2), (b, 2), \dots\} \approx A$
with $f: B \times A \rightarrow A$ defined by

$$f(a, i) = 2i$$

$$f(b, j) = 2j+1$$

where $i = 1, 2, 3, \dots$ and $j = 1, 2, 3, \dots$

Therefore, $2\omega = \text{ord}(B \times A) = \text{ord}(A)$

$$\Rightarrow 2\omega = \omega \quad \text{--- (2)}$$

From (1) and (2), we have

$$2\omega \neq \omega$$

Theorem.

For any ordinal numbers λ, μ, η

$$(i) \quad (\lambda\mu)\eta = \lambda(\mu\eta)$$

$$(ii) \quad \lambda(\mu+\eta) = \lambda\mu + \lambda\eta$$

$$(iii) \quad 1\lambda = \lambda 1 = \lambda$$

Proof.

~~$A_1 \neq \emptyset$~~

~~General mistake~~

~~$A_2 = \{1, 2, 3, 4\}$ $B = \{1, 2\}$~~
 ~~$f(A) = 2$~~
 ~~$g(A) = 3$~~

Cartesian Product.

Let $\{A_i : i \in I\}$ be a non empty collection of non empty sets. Then the cartesian product of $\{A_i : i \in I\}$ is the set of all choice functions defined on $\{A_i : i \in I\}$.

It is denoted by $\prod \{A_i : i \in I\}$ or $\prod A_i$.

Choice function.

Let A_i be a subset of X . Then choice function

$f : \{A_i : i \in I\} \rightarrow X$ is defined as

$$f(A_i) = a_i ; a_i \in A_i.$$

i.e. f chooses one point from each A_i .

Example.

In case of finite collection i.e. $\{A_1, A_2, \dots, A_n\}$ the cartesian product is

$$\prod A_i = \{f(A_1), f(A_2), \dots, f(A_n)\}$$

$$= \{(a_1, a_2, \dots, a_n)\}$$

Axiom of Choice. S/2013

The cartesian product of non empty collection of non empty sets is non empty.

OR: There exists a choice function for any non empty collection of non empty sets.

Zermelo's Postulate. S/2013

Let $\{A_i : i \in I\}$ be any non empty collection of disjoint non-empty sets. Then there exists

(135)

a subset B of the union $\bigcup A_i$ such that the intersection of B and each set A_i consists of exactly one element.

$$\text{i.e. } |B \cap A_i| = 1$$

Question. A/2012, S/2013, A/2016,

Show that the axiom of choice is equivalent to Zermelo's Postulate.

Solution.

Suppose that the cartesian product of non-empty collection of non empty sets is non-empty.

Let $\{A_i : i \in I\}$ be any non empty collection of disjoint non empty sets. Then by axiom of choice, there exists a choice function f on $\{A_i : i \in I\}$.

Consider the collection B such that

$$B = \{f(A_i) : i \in I\}.$$

Then clearly $B \cap A_i = \{f(A_i)\}$ consists of exactly one element as A_i are disjoint.

Thus axiom of choice implies Zermelo's postulate.

Conversely Suppose that Zermelo's postulate is satisfied.

Let $\{A_i : i \in I\}$ be a non empty collection of non empty sets.

$$\text{Put } A_i^* = A_i \times \{i\}, \quad \forall i \in I$$

Then clearly $\{A_i^*\}$ is a disjoint family of



136

non empty sets as

$$A_i \times \{i\} \cap A_j \times \{j\} = \emptyset \quad \text{for } i \neq j.$$

Then by Zermelo's postulate, there exists a

subset B of $\bigcup \{A_i : i \in I\}$ such that

$$B \cap A_i = \{a_i, i\}$$

consists of exactly one element. Then $a_i \in A_i$

and so the function f on $\{A_i : i \in I\}$ defined

by $f(A_i) = a_i$ is a choice function.

Thus Zermelo's postulate implies axiom of choice.

Zorn's Lemma. A/2016,

Let X be a non empty partially ordered set in which every chain has an upper bound in X .

Then X contains at least one maximal element.

Well-Ordering Theorem. A/2015,

Every non empty set can be well-ordered.

Proof.

Let X be a non empty set and let A be the collection of all well-ordered subsets of X .

If $w \in A$, then $w = (B_i, \leq)$ where $B_i \subseteq X$ and B_i is ordered by $\leq$.

Now define a partial order to A as follows:

if $w_1, w_2 \in A$ then $w_1 \leq w_2$ if either $w_1 \leq w_2$ or w_1 is an initial segment of w_2 .

$$(w_1 \leq w_2 \Rightarrow B_1 \subseteq B_2).$$

6

(137)

Let $\{w_i : i \in I\}$ be the totally ordered sub-collection of A . Then the collection $\{B_i : i \in I\}$ ordered by set inclusion is totally ordered.

Define $w = (B, \leq)$ as follows

$$B = \bigcup B_i$$

If $a, b \in B = \bigcup B_i$ then $a \in B_i, b \in B_j$ for some $i, j \in I$.

Since $\{B_i : i \in I\}$ is totally ordered by set inclusion so either $B_i \subseteq B_j$ or $B_j \subseteq B_i$.

Suppose $B_i \subseteq B_j$. Then $a, b \in B_j$.

$\Rightarrow a \leq b$ as elements of w if $a \leq b$ as element of B_j .

$\Rightarrow w = (B, \leq)$ is totally ordered.

$\Rightarrow w \in \{w_i : i \in I\}$.

Now as $B_i \subseteq \bigcup B_i$

$\Rightarrow B_i \subseteq B$ by ①

$\Rightarrow w_i \leq w \quad \forall i \in I$

$\Rightarrow w$ is an upper bound of $\{w_i : i \in I\}$.

$\Rightarrow$ Every totally ordered sub-collection of A has an upper bound.

Then by Zorn's Lemma A has at least one maximal element say w^* .

We claim that $w^* = X$. For if $w^* \neq X$ then there exists $a \in X$ such that $a \in X \setminus w^*$.

-- 1 is uncountable

(138)

Now consider the set $P = \{w^*, \{a\}\}$. Then

P is well-ordered. Hence $P \in A$.

Also w^* is an initial segment of P .

$$\Rightarrow w^* < P.$$

which contradicts the assumption that w^* is maximal element of A .

$$\text{Hence } w^* = X.$$

$\Rightarrow X$ can be well-ordered.

A/2018

Q4(b).

First we show that set A of algebraic numbers is denumerable.

As we know that the set E of all polynomials equations is denumerable.

$$E = \{p_1(x)=0, p_2(x)=0, \dots\}$$

Define

$$A_k = \{x : x \text{ is a solution of } p_k(x)=0\}.$$

Since a polynomial of degree n can have at most n roots,

139

each A_k is finite.

Thus $A = \bigcup \{A_k : k \in \mathbb{P}\}$ is a countable family of countable sets.

Accordingly, A is countable set and A is not finite.

So A is denumerable.

Now we show that the set T of transcendental number is uncountable.

First we define the transcendental number.

A real number x is called transcendental number if x is not algebraic.

i.e. x is not solution to a polynomial equation

$$p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n = 0$$
 with integral co-efficients.

By definition it is clear that transcendental number are real numbers not algebraic.

Since set of real number is uncountable so T is uncountable.

A/2018, A/2012, A/2014

140

Theorem.

Let X be a partially ordered set. Then X contains a maximal chain.

Proof.

Let X be a non empty poset and A be the collection of all chains in X .

i.e. $A = \{C : C \text{ is a chain in } X\}$
where A is poset w.r.t set inclusion ordering.

Let B be a linearly ordered collection of sets in A .

i.e. $B = \{C_i : C_i \text{ is a chain in } X\}$.

Now define $D = \bigcup C_i$.

We need to prove that D is chain in X or $D \in A$.

For this, let $a, b \in D = \bigcup C_i$. Then $a \in C_k$ and $b \in C_j$. Since $B = \{C_i : i \in I\}$ be linearly ordered.

So either $C_k \subseteq C_j$ or $C_j \subseteq C_k$.

If $C_k \subseteq C_j$ then $a \in C_j$.

141

Since C_j is a chain so a, b are comparable.

$\Rightarrow D$ is a chain or $D \in A$.

Also D being union of all chains is an upper bound of set B .

Then by Zorn lemma A has a maximal chain in X .

Theorem. A/2018. A/2015

Suppose that A is any uncountable set B is a denumerable subset of A . Show that $A \setminus B \approx A$.

Proof.

Let A be an uncountable set and $B = \{b_1, b_2, \dots\}$ be a denumerable subset of A .

The set $A \setminus B$ is infinite and contains a denumerable subset,

say, $D = \{d_1, d_2, d_3, \dots\}$.

Let $A^* = A \setminus (B \cup D)$. Then A and $A \setminus B$ are the following disjoint unions,

$$A = A^* \cup D \cup B$$

$$= A^* \cup \{d_1, d_2, d_3, \dots\} \cup \{b_1, b_2, b_3, \dots\}$$

$$A \setminus B = A^* \cup D = A^* \cup \{d_1, d_2, d_3, \dots\}$$

Define $f: A \rightarrow A/B$ as

$$f(a) = a \quad \text{if } a \in A^*$$

$$f(d_n) = d_{2n-1}, \quad n \in \mathbb{P}$$

$$f(b_n) = d_{2n}, \quad n \in \mathbb{P}$$

$$A = A^* \cup \{d_1, d_2, d_3, \dots\} \cup \{b_1, b_2, b_3, \dots\}$$

$$A/B = A^* \cup \{d_1, d_2, d_3, d_4, d_5, d_6, \dots\}$$

Then clearly from Fig, f is
~~an~~ one-one and onto.

Hence $A/B \cong A$.